Probability and Random Processes
R.J. Marks II Lecture Notes
University of Washington (1984)
TIE PROGRAM                                  Summer Quarter 1984

EE 505 - Introduction to Probability and Random Processes
         4 credits

Professor Robert Marks

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<th>Time</th>
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<td>12:00 - 1:00</td>
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List of students in the Televised Instruction in Engineering Program who are enrolled in this course, their telephone number, and their company affiliation.

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<th>Student</th>
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<tbody>
<tr>
<td>1. Baruah, Arati Bora</td>
<td>Boeing Computer</td>
<td>656-5741</td>
<td>271-1773</td>
</tr>
<tr>
<td>2. Brown, Marc Gerard</td>
<td>Hewlett-Packard</td>
<td>335-2028</td>
<td>568-1683</td>
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<td>4. Corulli, Charles</td>
<td>Fairchild</td>
<td>841-6022</td>
<td>858-7075</td>
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<td>5. Ellersick, Steven Donald</td>
<td>Boeing Military</td>
<td>394-4268</td>
<td>324-9125</td>
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<td>6. Fitzsimmons, Dan Kenneth</td>
<td>Boeing Aerospace</td>
<td>251-0238</td>
<td>838-7185</td>
</tr>
<tr>
<td>7. Lavering, Laura Jean</td>
<td>Boeing Computer</td>
<td>O:</td>
<td>H:</td>
</tr>
<tr>
<td>10. Simon, Daniel John</td>
<td>Boeing Aerospace</td>
<td>773-8033</td>
<td>392-8156</td>
</tr>
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</table>
INSTRUCTIONS:

- Monday, July 15, 1996; 2:20 PM to 4:20 PM.
- Write your name on the upper right hand side of this sheet.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single legal sized sheet of notes and calculator.
- Each problem is worth 20 points.
- TIE students must identify the exam proctor and have the proctor initial the examination.

1. Specify whether the following pairs of events are (a) independent, (b) mutually exclusive or (c) neither.

- A balanced budget amendment bill passes congress by August 15. A balanced budget amendment does not pass congress by August 15. **mutually exclusive**
- The sum on two dice is seven. There are six dots on the first die. **Ind.**
- You have an ace in your poker hand. Your opponent has an ace in their poker hand. **neither**
- You win the Washington state lottery. Your mother wins the New York lottery. **Ind.**
- You receive one call before noon. You receive two calls all day. **neither**

---

1Four points for a correct answer, zero for no answer and -2 for an incorrect answer.
2. Ken Griffey Jr. has a batting average of 0.300. Assume this means, each time he bats, his probability of getting a hit is 0.300. Estimate the probability that he gets 300 or more hits in his next 1000 at bats.

\[
p = 300; n = 1000
\]

\[
P_r[k \geq 300] = \sum_{k=300}^{1000} \binom{n}{k} p^k q^{n-k}
\]

\[
np q = 0.300 \times 1000 \times 0.70 = 210
\]

\[
P_r[k \geq 300] = P_r[k < 300] \approx G \left( \frac{300 - np}{\sqrt{npq}} \right)
\]

\[
= G(0) = \frac{1}{2}
\]
3. In Lake Wobegon, there are 10,000 catfish, 20,000 perch and 30,000 dogfish. The dogfish are very hungry and are twice as likely to be caught. Bobby Jack went fishing and caught four fish. The probability that three were dogfish and one was a perch were caught can be written as

\[
\frac{2^p}{3^q}.
\]

What are the integers \( P \) and \( Q \)?

\[
\begin{align*}
p_1 &= \frac{10^k}{10^k + 20k + 2 \times 30k} \\
p_2 &= \frac{2}{9} \\
p_3 &= \frac{6}{9} = \frac{2}{3} \\
k_1 &= 0 \\
k_2 &= 1 \\
k_3 &= 3
\end{align*}
\]

Generalized Bernoulli trial:

\[
P_r[k_1, k_2, k_3] = \frac{n!}{k_1!k_2!k_3!} \cdot p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3}
\]

\[
P_r[0, 1, 3] = \frac{4!}{0!1!3!} \cdot \left(\frac{1}{9}\right)^0 \cdot \left(\frac{2}{9}\right)^1 \cdot \left(\frac{2}{3}\right)^3
\]

\[
= 4 \cdot \frac{2^3}{3^2} \cdot \frac{2^3}{3^3} = \frac{2^6}{3^5}
\]

\[
p = 6, \quad q = 5
\]

\[
\left(\frac{2^6}{3^5} = 0.263\right)
\]
4. Bill eats only Big Macs and sausage pizzas. Big Macs give him heartburn 10% of the time. The pizza’s give him heartburn 20% of the time. He eats twice as many pizzas as Big Macs. Bill has heartburn. What is the probability it was caused by a Big Mac?

\[ P[H] = P[H/\text{BM}]P[\text{BM}] + P[H/\text{Pizza}]P[\text{Pizza}] \]

\[ = \frac{1}{10} \times \frac{1}{3} + \frac{2}{10} \times \frac{2}{3} = \frac{5}{30} = \frac{1}{6} \]

\[ P_r[\text{BM}/H] = \frac{P_r[H/\text{BM}]P_r[\text{BM}]}{P[H]} \]

\[ = \frac{\frac{1}{10} \times \frac{1}{3}}{\frac{1}{6}} = \frac{6}{30} = \frac{1}{5} = 0.20 \]
5. A Poisson random variable with parameter $\lambda = 2$ occurrences per hour is observed for a half hour. The probability that the number of occurrences exceeds or is equal to two given that the total number of occurrences exceeds or equals one can be written as

$$\frac{1 - a e^c}{1 - b e^d}$$

Identify the numbers $a$, $b$, $c$ and $d$.

$$\lambda T = 2 \times \frac{1}{2} = 1$$

$$\mathcal{P} = \Pr[X \geq 2 | X \geq 1] = \frac{\Pr[X \geq 2, \text{and } X \geq 1]}{\Pr[X \geq 1]}$$

$$= \frac{\Pr[X \geq 2]}{\Pr[X \geq 1]}$$

$$= \frac{1 - [\Pr(X = 0) + \Pr(X = 1)]}{1 - \Pr(X = 0)}$$

$$\Pr[X = k] = \frac{(\lambda T)^k}{k!} e^{-\lambda T} = \frac{1}{k!} e^1 = \frac{e^{-1}}{k!}$$

Thus

$$\Pr[X = 0] = e^{-1} \quad \text{and} \quad \Pr[X = 1] = e^{-1}$$

and

$$\mathcal{P} = \frac{1 - (e^{-1} + e^{-1})}{1 - e^{-1}} = \frac{1 - 2e^{-1}}{1 - e^{-1}}$$

Thus: $a = 2$, $b = c = d = 1$
6. Washington state apples are modeled with a Gaussian pdf. If $X$ is the
diameter,

$$X \sim N(\mu = 3, \sigma = 2)$$

Apples below two inches in diameter and above four inches are discarded.
What is the probability that an apple passing this test is three inches or less
in diameter?

$$P = P_r \left[ \frac{X \leq 3}{2 \leq X \leq 4} \right]$$

$$= \frac{P_r \left[ X \leq 3, 2 \leq X \leq 4 \right]}{P_r \left[ 2 \leq X \leq 4 \right]}$$

Now, consider $X_1$ and $X_2$ as the diameter.

Recall:

$$P_r \left[ x_1 \leq X \leq x_2 \right] = G \left( \frac{x_2 - \mu}{\sigma} \right) - G \left( \frac{x_1 - \mu}{\sigma} \right)$$

Thus

$$P = \frac{\text{erf} \left( \frac{3 - 3}{2} \right) - \text{erf} \left( \frac{2 - 3}{2} \right)}{\text{erf} \left( \frac{4 - 3}{2} \right) - \text{erf} \left( \frac{2 - 3}{2} \right)}$$

$$= \frac{\text{erf}(0) - \text{erf}(-\frac{1}{2})}{\text{erf}(\frac{1}{2}) - \text{erf}(-\frac{1}{2})} = \frac{\text{erf}(\frac{1}{2})}{2 \text{ erf}(\frac{1}{2})} = \frac{1}{2}$$

Of course!

![Graph of $f_X(x)$ from $2$ to $3$ and $4$ with $P_r[X \leq 3 | 2 \leq X \leq 4]$]
7. Matlab's error function is

\[ \text{erf}_{ML}(x) = \frac{2}{\sqrt{\pi}} \int_{t=0}^{x} e^{-t^2} dt \]

Papoulis' definition is

\[ \text{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{z=0}^{y} e^{-z^2} dz \]

We wish to find \( \text{erf}(2) \) using Matlab. How do you do it?

\[
\begin{align*}
\text{erf}(y) & = \frac{1}{\sqrt{2\pi}} \int_{0}^{y} e^{-z^2} \, dz \\
& = \frac{1}{\sqrt{2\pi}} \int_{t=0}^{y/\sqrt{2}} e^{-t^2} \, (\sqrt{2} \, dt) \\
& = \frac{1}{\sqrt{\pi}} \int_{0}^{y/\sqrt{2}} e^{-t^2} \, dt \\
& = \frac{1}{2} \left[ \frac{2}{\sqrt{\pi}} \int_{0}^{y/\sqrt{2}} e^{-t^2} \, dt \right] \\
& = \frac{1}{2} \text{erf}_{ML}(y/\sqrt{2})
\end{align*}
\]

Thus:

\[ \text{erf}(2) = \frac{1}{2} \text{erf}_{ML}(\frac{2}{\sqrt{2}}) = \frac{1}{2} \text{erf}_{ML}(\sqrt{2}) \]
INSTRUCTIONS:

- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single legal sized sheet of notes and calculator.
- Each problem is worth 20 points.
- The Error Function Table is on Page 8 of this booklet.

1. Specify whether the following pairs of events are (a) independent, (b) mutually exclusive or (c) neither.

- A health care bill passes congress by August 15. A health care bill does not pass congress by August 15. (B)
- The sum on two dice is seven. There are six dots on the first die. (A)
- You have an ace in your poker hand. Your opponent has an ace in their poker hand. (C)
- You win the Washington state lottery. Your mother wins the New York lottery. (A)
- You receive one call before noon. You receive two calls all day. (C)

1. Let $A = \text{bill passes}$, $B = \text{bill does not pass}$.
Then $B = A^c$ : Mutually exclusive

2. $P[\text{Sum is 7}] = \frac{1}{6} = P[A]$
$P[6 \text{ on 1st die}] = \frac{1}{6} = P[B]$
$P[\text{Sum is 7 and 6 on (1st die)}] = P[6,1] = \frac{1}{36} = P[AB]$

3. $P[A] = \frac{4}{52}$, $P[B] = \frac{4}{52}$, $P[AB] = P[A]P[B] + P[\overline{A}B] \neq 0$ : (C)


5. $P[\overline{A}B] \neq P[A]P[\overline{B}]$, $P[AB] \neq 0$ : (C)

1 Four points for a correct answer, zero for no answer and -2 for an incorrect answer.
2. Ken Griffey Jr. has a batting average of 0.300. Assume this means, each time he bats, his probability of getting a hit is 0.300. Estimate the probability that he gets 850 or more hits in his next 1000 at bats.

\[ p = 0.3, \quad q = 0.7, \quad n = 1000, \quad np = 300 \]

\[ P \left( \theta \geq 850 \right) = P \left( \theta \geq 850 \right) \]

\[ = \sum_{k=850}^{1000} \binom{n}{k} p^k q^{n-k} \]

\[ = G \left( \frac{1000 - np}{\sqrt{npq}} \right) - G \left( \frac{850 - np}{\sqrt{npq}} \right) \]

\[ = G \left( \frac{700}{14.5} \right) - G \left( \frac{550}{14.5} \right) \]

\[ = G (49.2) - G (37.9) \]

\[ \approx 0 \]
3. In Lake Wobegon, there are 10,000 catfish, 20,000 perch and 30,000 dogfish. The dogfish are very hungry and are twice as likely to be caught. Bobby Jack went fishing and caught four fish. What is the probability that three were dogfish and one was a perch?

Let
\[ A = \{ \text{Catch Dogfish} \} \]
\[ B = \{ \text{Catch Perch} \} \]
\[ C = \{ \text{Catch Catfish} \} \]

Then
\[ p(A) = \frac{6}{9} = \rho \]
\[ p(B) = \frac{3}{9} = \eta \]
\[ p(C) = \frac{1}{9} = \tau \]

\[ P_4(3,1) = \frac{4!}{3! \cdot 1!} \left( \frac{6}{9} \right)^3 \left( \frac{2}{3} \right)^1 \]
\[ \approx 0.26 \]
4. Bill eats only Big Macs and sausage pizzas. Bic Macs give him heartburn 10% of the time. The pizza's give him heartburn 20% of the time. He eats twice as many pizzas as Big Macs. Bill has heartburn. What is the probability it was caused by a Big Mac?

\[ \text{HB = Heartburn, BM = Big Mac, PI = Pizza} \]

\[ P(\text{HB}) = P(\text{HB, BM}) + P(\text{HB, PI}) \]

\[ = \frac{P(\text{HB|BM})P(\text{BM})}{0.1} + \frac{P(\text{HB|PI})P(\text{PI})}{0.2} \]

\[ = 0.1 \times \frac{1}{3} + 0.2 \times \frac{2}{3} \]

\[ = \frac{0.5}{3} \]

\[ P(\text{BM|HB}) = \frac{P(\text{HB, BM})}{P(\text{HB})} \]

\[ = \frac{\frac{0.1}{3}}{\frac{0.5}{3}} \]

\[ = \frac{1}{5} \]
5. A Poisson random variable with parameter \( \lambda = 2 \) occurrences per hour is observed for a half hour. What is the probability that the number of occurrences exceeds two given that the total number of occurrences exceeds one?

Let \( A = \{ \text{number of occurrences} > 2 \} \)

\[ B = \{ \text{total number of occurrences} > 1 \} \]

Clearly \( B \supseteq A \)

\[
\Pr(A^c \cap B) = \Pr(k > 2) = 1 - \Pr(0 \leq k \leq 2)
\]

\[
= 1 - \left[ \Pr(k = 0) + \Pr(k = 1) + \Pr(k = 2) \right] = 1 - \left[ e^{-1} \left( \frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!} \right) \right]
\]

\[
\approx 0.08
\]

\[
\Pr(B^c) = \Pr(k > 1) = 1 - \Pr(k = 0) - \Pr(k = 1)
\]

\[
= 1 - \left[ e^{-1} (1) + e^{-1} (1) \right] = 0.26
\]

\[
\Pr(k > 2 \mid k > 1) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} = \frac{0.08}{0.26}
\]

\[
\approx 0.30
\]
6. Washington state apples are modeled with a Gaussian pdf. If $X$ is the diameter,

$$X \sim N(\eta = 3, \sigma = 1)$$

Apples below two inches in diameter and above four inches are discarded. What is the probability that an apple, after the discarding, is three inches or less in diameter?
7. Let $Y$ be a Bernoulli trial with probability of success $p$. We perform the Bernoulli trial until we get a success. Let $N$ denote the number of trials needed to achieve a success. What is the pdf of $N$?

We just need one success. Then we stop.

If $n$ is a trials, this means $n$ failures, success at $n$th trial.

$$f_N(n) = \binom{n-1}{0} p^0 q^{n-1} \times q$$

$$= p \cdot (1-p)^{n-1}, \quad n \geq 1$$

Check: $\sum_{n=1}^{\infty} f_N(n) = 1$

$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = p \sum_{n=1}^{\infty} (1-p)^{n-1}$$

$$= p \cdot \frac{1}{1 - (1-p)} = \frac{p}{p} = 1.$$
SOME CORRECTIONS ON EE505 TEXT (PAPOUNIS)

1. Page 82, Eq. (4-49)

\[
\frac{ce^{-cx}}{ce^{-ct}} = e^{-c(x-t)}
\]

should be

\[
\frac{ce^{-cx}}{e^{-ct}} = ce^{-c(x-t)}
\]

2. Page 75, Eq. (4-32)

\[
\delta(n-k) \text{ should be } \delta(x-k)
\]

3. Page 92, Ex. 5-3, second sentence

\[
F_X(-c) - F_X(c) \text{ should be } F_X(c) - F_X(-c)
\]

4. Page 148, Problem 6-18

Missing right paren

5. Page 171, Problem 7-10

\[
E\{U(a-x)\} E\{U(a-y)\} \text{ should be } E\{U(a-x)\} E\{U(b-y)\}
\]

6. Page 132, Two lines above second equation

\[
x < b \cos \theta \text{ should be } x < a \cos \theta
\]

7. Page 132, Second equation

Upper integration limit should be \(\pi/2\) instead of \(a\)

8. Page 195, First equation

\[
\eta_i = T \text{ should be } T/2
\]

Next line, \(\eta = 2T\) should be \(\eta = T\)
   The two cosine terms should be added, not subtracted

10. Page 213, Eq. 9-20
    \[ 2^k \] should be \[ 2^n \]

11. Page 232, Third equation
    The product \( x_1 x_2 \) should be deleted

12. Page 224, Last equation
    The \( n=0 \) and \( n\neq0 \) should be interchanged

13. Page 252, Last equation
    Bracket and semicolon are missing

14. Page 259, Problem 9-4, third line
    \( y \) in \( y(t) \) should be bold faced

15. Page 261, Problem 9-30
    "...where \( x(t) = \)" should be "...where \( R_X(\tau) = \)"

16. Page 572, Index
    Gamma density entry should list page 77
17. Page 9, Example 1-3, third line
   \[ e \sqrt{3} \] should be \( r \sqrt{3} \)

18. Page 60, last line before problems
   \( r \) should be \( p_r \)

19. Page 76, second line
   46 should be 40

20. Page 93, Ex. 5-6, second equation
    missing right paren

21. Page 100, one line after 5-14
    \( x \) should be \( y \)

22. Page 122, problem 5-27
    \( 2q/p^2 \) should be \( q/p^2 \)

23. Page 120, Fig. 6-5b
    \( x \) should be \( x_3 \)

24. Page 141, Ex. 6-10, first equation
    second \( x(f_1 f_k) \) should be \( y(f_1 f_k) \)

25. Page 154, Ex. 7-2

\[
\frac{\tau^2_{zw}}{\sigma^2_z \sigma^2_w} = \frac{[E(z^w) - E(z)E(w)]^2}{\sigma^2_z \sigma^2_w} = \frac{9}{7x3}
\]
26. Page 154, Ex. 7-2, Last equation should be
\[ N(10,10; \sqrt{7}, \sqrt{3}; \sqrt{3/7}) \]

27. Page 171, Problem 7-15
\[ E([y-g(x)]^2) \]

28. Page 187, last equation
missing \( d\chi \)

29. Page 192
(a) line above fourth equation
\[ p(1-q) \text{ should be } p(1-p) \]
(b) fourth equation
second inequality should be reversed

30. Page 198, first equation
denominator should be \( \sqrt{n} \sigma \) instead of \( \sigma \)

31. Page 212, line above Eq. (9-19)
"of" should be "or"

32. Page 213, Eq. 9-24
\[ f(w,t) = \]

33. Page 260, Prob. 9-16
34. Page 266, Eq. 10-13
\[ G \left( \frac{a}{\sqrt{-R_{XX}(0)}} \right) \]
delete \( 1/2\pi \)

Acknowledgements: 17-33 were detected by Peter Wai
Work

\[(3-1) \Rightarrow (p-q)^n -(p-q)^n = \text{even}\]

\[
\binom{n}{k} = \text{# orderings k good}
\]

\[
\binom{n-k}{n-k} = \text{# n-k bad}
\]

\[
\binom{k}{n-k} \Rightarrow \# \text{ k good} + n-k \text{ bad}
\]

4. \(n = 900\)

\[
\sigma = npq = 900 = 225
\]

\[
\epsilon = \sqrt{n} \frac{p-1}{\sigma} = 420 - \text{erf} \left( \frac{k-np}{\sigma} \right) - \epsilon(\frac{k-np}{\sigma})
\]

\[
\text{erf} (920)
\]

7. \(p = 1 - e^{-r/\sqrt{T}} = 1 - e^{-4}\)

\[
npq >> 1 \Leftrightarrow \text{DeMoivre} npq
\]

\[
\text{and } (n-k) \text{ tails}
\]

12. k-1 heads at n-1

\[
\binom{n-1}{k-1} \text{ p } \cdot \text{ q } \cdot \text{ (n-k) tails)
\]

14. Russian Roulette

\[
P(A) = P(A | M) P(M) + P(A | \overline{M}) P(\overline{M})
\]

\[
p = 1 \times \frac{2}{36} + (1-p) \frac{34}{36}
\]

\[
p \left(1 + \frac{34}{36}\right) = \frac{2}{36}
\]

\[
p \left( \frac{70}{36} \right) = \frac{2}{36} \frac{36}{70} = \frac{13}{35}
\]
1. For \( \bar{Y} = \frac{1}{N} \sum_{n=1}^{N} X_n \)

if all the \( X_n \)'s are independent:

Thus:

\[
\Phi_{\bar{Y}}(\omega) = \Phi_{\bar{X}}^{N} (\frac{\omega}{N})
\]

\[
\frac{d}{d\omega} \Phi_{\bar{X}}(\omega) = \frac{d}{d\omega} \Phi_{\bar{X}}^{N} (\frac{\omega}{N}) = \Phi_{\bar{X}}^{N-1} (\frac{\omega}{N}) \Phi'_{\bar{X}} (\frac{\omega}{N})
\]

\[
\Rightarrow \frac{d}{d\omega} \Phi_{\bar{Y}}(\omega) = \Phi_{\bar{X}}^{N-1} (\omega) \Phi'_{\bar{X}} (\omega) = \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

Now:

\[
\frac{d}{d\omega} \Phi_{\bar{X}}(\omega) = \frac{d}{d\omega} \Phi_{\bar{X}}^{N-1} (\omega) \Phi'_{\bar{X}} (\omega)
\]

\[
= \frac{N-1}{N} \Phi_{\bar{X}}^{N-2} (\omega) \left[ \Phi'_{\bar{X}} (\omega) \right]^2 + \frac{N}{N} \Phi_{\bar{X}}^{N-1} (\omega) \Phi''_{\bar{X}} (\omega)
\]

\[
\Rightarrow \frac{d}{d\omega} \Phi_{\bar{X}}(\omega) = \frac{N-1}{N} \Phi_{\bar{X}}^{N-2} (\omega) \left[ \Phi'_{\bar{X}} (\omega) \right]^2 + \frac{N}{N} \Phi_{\bar{X}}^{N-1} (\omega) \Phi''_{\bar{X}} (\omega)
\]

\[
\frac{d}{d\omega} \Phi_{\bar{X}}(\omega) = \frac{N-1}{N} \Phi_{\bar{X}}^{N-2} (\omega) \left[ \Phi'_{\bar{X}} (\omega) \right]^2 + \frac{N}{N} \Phi_{\bar{X}}^{N-1} (\omega) \Phi''_{\bar{X}} (\omega)
\]

\[
\Rightarrow \variance \bar{Y} = \frac{d}{d\omega} \Phi_{\bar{X}}(\omega) = \frac{N}{N} \Phi_{\bar{X}}^{N-1} (\omega) \Phi''_{\bar{X}} (\omega)
\]

2. For Cauchy:

\[
f_{\bar{X}}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}
\]

From last homework:

\[
\Phi_{\bar{X}}(\omega) = e^{-\alpha |\omega|}
\]

\[
\Phi_{\bar{X}}(\omega)
\]

\[
\frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

\[
\Rightarrow \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

To get variance, we need

\[
\frac{\alpha}{\pi} \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

But \( \frac{d}{d\omega} \Phi_{\bar{X}}(\omega) \) has a delta function at the origin and \( \frac{x^2}{x^2} = 0 \).

\[
\variance \bar{X} = \frac{\alpha}{\pi} \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

\[
\Rightarrow \variance \bar{X} = \frac{\alpha}{\pi} \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

\[
\Rightarrow \variance \bar{X} = \frac{\alpha}{\pi} \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]

\[
\Rightarrow \variance \bar{X} = \frac{\alpha}{\pi} \frac{d}{d\omega} \Phi_{\bar{X}}(\omega)
\]
Find the density, mean and variance of

\[ Y = \frac{1}{N} \sum_{n=1}^{N} X_n \]

when \( X \) is

(a) \text{Poisson} \quad \text{and} \quad \text{b) Gamma}, \quad \text{b} = n

In general:

\[ \Phi_Y(\omega) = \Phi_X^N\left(\frac{\omega}{N}\right) \]

(a) For Poisson:

\[ \Phi_Y(\omega) = e^a(e^{i\omega} - 1) \]

Thus:

\[ \Phi_Y(\omega) = e^{a e^{i\omega} - 1} \]  \hspace{1cm} (1)

Since \( e^b \sum_{n=0}^{\infty} \frac{b^n}{n!} \delta(x-n) \) Fourier transforms to \( \exp[b(e^{i\omega} - 1)] \) then, from the scaling theorem, (1) inverse transforms to:

\[ f_Y(x) = N e^{-aN} \sum_{n=0}^{\infty} \frac{(aN)^n}{n!} \delta(Nx-n) \]

\[ = e^{-aN} \sum_{n=0}^{\infty} \frac{(aN)^n}{n!} \delta(x-n) \]

\[ \bar{y} = \int_{-\infty}^{\infty} y f_Y(y) dy \]

\[ = e^{-aN} \sum_{n=0}^{\infty} \frac{(aN)^n}{n!} \]

set \( m = n - 1 \)

\[ \Rightarrow \bar{y} = \frac{1}{N} e^{-aN} \sum_{m=0}^{\infty} \frac{(aN)^{m+1}}{m!} \]

\[ = \frac{1}{N} e^{-aN} (aN) e^{aN} \]

\[ = a = \bar{x} \]
\[
\overline{y}^2 = e^{-aN} \sum_{n=0}^{\infty} \left( \frac{n}{N} \right)^2 \frac{(aN)^n}{n!} = e^{-aN} \frac{1}{N^2} \sum_{n=1}^{\infty} \frac{n(aN)^n}{(n-1)!}
\]

but:
\[
\int \frac{n(aN)^{n-1}}{(n-1)!} \, d(aN) = \sum_{n=1}^{\infty} \frac{(aN)^n}{(n-1)!} = (aN) e^{aN}
\]

thus:
\[
\sum_{n=1}^{\infty} \frac{n(aN)^{n-1}}{(n-1)!} = (1 + aN) e^{aN}
\]

\[
\overline{y}^2 = e^{-aN} \frac{1}{N^2} \quad aN \left[1 + aN \right] e^{aN}
\]

Thus:
\[
\sigma_y^2 = \overline{y}^2 - \overline{y}^2 = \frac{a(1 + aN)}{N} - a^2
\]

\[
= \frac{a}{N} + a^2 - a^2 = \frac{a}{N} = \text{var} \, x
\]

(b) For GAMMA:
\[
f_X(x) = \frac{c^{b+1}}{\Gamma(b+1)} x^b e^{-cx} \mu(x)
\]

from (5.72) on p. 159:
\[
\Phi_X(\omega) = \frac{c^{n+1}}{n!} x^n e^{-cx} \mu(x)
\]

Thus:
\[
\Phi_Y(\omega) = \frac{c^{N(n+1)}}{(c - j\omega)^{N(n+1)}}
\]

For \(m+1 = N(n+1)\), this inverse transforms to:
\[
f_X(x) = N \frac{c^{n+1}}{m!} (NX)^m e^{-(NX)} \mu(x)
\]

or:
\[
f_X(x) = \left[ \frac{Nc^{N(n+1)}}{N(n+1)-1} \right]^{(N(n+1)-1)} (NX)^m e^{-(NX)} \mu(x)
\]
or:

\[ f_\bar{x}(x) = \frac{(NC)^{N(n+1)}}{[N(n+1)-1]!} x^{N(n+1)-1} e^{-(NC)x} \mu(x) \]

This is a gamma density with parameters:

\[ \hat{b} = N(n+1) \]
\[ \hat{c} = NC \]

The mean is thus: (pg 147)

\[ \bar{y} = \frac{\hat{b} + 1}{\hat{c}} = \frac{n+1}{c} = \overline{x} \]
\[ \text{var } \bar{x} = \frac{\hat{b} + 1}{\hat{c}^2} = \frac{n+1}{NC^2} = \frac{\text{var } x}{N} \]

Here, as in all previous examples,
\[ \bar{y} = \overline{x} \quad \text{and} \quad \text{var } \bar{x} = \frac{1}{N} \text{ var } x \]

Homework:

1. For \( \overline{X} = \frac{1}{N} \sum_{n=1}^{N} X_n \), show:

\[ \bar{y} = \overline{x} \quad \text{var } \overline{X} = \frac{1}{N} \text{ var } X \]

assuming everything exists.

2. For the Cauchy distribution, discuss what happens when we attempt to evaluate the mean and variance by differentiating the characteristic function.

Also: Chapt 6: 2, 4, 6, 7, 8
Chapt 7: 1, 2, 3, 4, 5
Solutions to midterm #2

\[ z = (-1)^{x + y} \]

\[ E[z] = E[(−1)^{x + y}] = E[(-1)^x] \cdot E[(-1)^y] \]

\[ (−1)^x = \sum_{n=0}^{\infty} \frac{e^{na} - e^{-na}}{n!} = e^{a} \sum_{n=0}^{\infty} \frac{(−a)^n}{n!} = e^{-2a} \]

Therefore: \[ E[z] = \exp[-2(a + b)] \]

\[ E[z^2] = E[(−1)^{x + y}]^2 = 1 \]

\[ \therefore \text{var} z = 1 - \exp[-4(a + b)] \]

\[ f_{X}(x) = \int_{-\infty}^{\infty} f(x, y) dy = 2 \int_{y:0}^{\sqrt{a^2-x^2}} dy/\pi a^2 = \frac{2\sqrt{a^2-x^2}}{\pi a^2}, \quad |x| \leq a \]

\[ : f(y/x) = f(x, y)/f_{X}(x) \]

\[ = \frac{4/\pi a^2}{2\pi a^2\sqrt{a^2-x^2}} = \frac{1}{2(\pi a^2)^2} \int_{|x| \leq a} \left( \sqrt{a^2-x^2} \right)^{-1} \]

and \[ E[Y/X] = \int_{-\infty}^{\infty} y \cdot f(y/x) dy = 0 \]

(this is because \( f(y/x) \) is not explicitly a function of \( y \) so the integration limits are over \( \sqrt{a^2-x^2} \leq y \leq \sqrt{a^2+y^2} \)

\[ \text{var} Y/X = E[Y^2/X] = \int_{-\infty}^{\infty} y^2 \cdot f(y/x) dx \]

\[ = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 \frac{dy}{2a^2\sqrt{a^2-x^2}} \]

\[ = \frac{1}{\pi a^3a^2-x^2} \cdot \int_{0}^{\sqrt{a^2-x^2}} y^2 dy^2 \]

\[ = \frac{1}{\pi a^3a^2-x^2} \cdot \frac{1}{3} (a^2-x^2)^{3/2} \]

\[ = \frac{a^2-x^2}{3\pi a^6}, \quad r < a \]

This is minimum when \( x = \pm a \). Then, we must require that \( Y = 0 \) and \( \text{var} Y/X = 0 \). This makes sense!
\[ g_n = \sum_{k=1}^{N} t_{nk} f_k \]
\[ E g_n = \sum_{k=1}^{N} t_{nk} E f_k = 0 \]
\[ E g_n^2 = \text{var}(g_n) = E \left[ (\sum_{k=1}^{N} t_{nk} f_k)^2 \right] \]
\[ = E \left[ \sum_{k=1}^{N} \sum_{l=1}^{N} t_{nk} t_{nl} f_k f_l \right] \]
\[ = \sum_{k=1}^{N} \sum_{l=1}^{N} t_{nk} t_{nl} \sigma^2 \delta_{k-n} = \sigma^2 \sum_{k=1}^{N} (t_{nk})^2 \]
\[ = 1 \]
\[ P_r [ |g_n| \leq 1 ] = 2 \text{erf} (1) \text{ by CLT} \]

\[ z(t) = e^{j \pi t} \]
\[ E z(t) = E e^{j \pi t} = e^{-\sigma^2 t^2/2} e^{j n t} = \mathcal{N}_z(t) \]
(follows from def of characteristic function for a normal r.v.).

\[ E z(t_1) z^*(t_2) = R(t_1, t_2) \]
\[ = E e^{j \pi (t_1 - t_2)} \]
\[ = e^{-\sigma^2 (t_1 - t_2)^2/2} e^{j n (t_1 - t_2)} \]

or

\[ R(t) = e^{-\sigma^2 t^2/2} e^{j n t} \]

Then

\[ \text{var} z(t) = R(0) = 1 \]

The process is not stationary since

\[ \mathcal{N}_z(t) \neq \text{constant} \]
modulo $2\pi$ moves masses

add to get:

Uniform on $-\pi$ to $\pi$. 
How can we generate

\[ A: \quad \mathcal{Y} = \mathcal{X} + \frac{1}{2} \]

What about a dice roll?

\[ D_n = \text{Int}\left(6X_n + 1\right) \]

How about the sum of two dice?

\[ D_n + D_{n+1} = \text{sum} \]

Gaussian R.V.

(a) can find the \( \mathcal{Y} \in \mathcal{Y}_n = g(X_n) \)

is gaussian (ugly)

(b) Central limit theorem

\[ \mathcal{Y} = X_1 + X_2 + X_3 + \ldots + X_N \]

\[ \mathcal{Y} \approx \text{Mean} = \frac{N}{2}, \quad \text{var} = \frac{N}{12} \]

(c) Compute \( X_n \frac{1}{2} X_{n+1} \)

then

\[ \mathcal{Y}_n = (-2 \ln X_1)^{\frac{1}{2}} \cos 2\pi X_1 \]

\[ \mathcal{Y}_{n+1} = (-2 \ln X_1)^{\frac{1}{2}} \sin 2\pi X_2 \]

\[ \mathcal{Y}_n \frac{1}{2} \mathcal{Y}_{n+1} \text{ are zero mean unit variance normal r.v.'s.} \]
Generating Random Numbers

Use Table

Pseudo-Random numbers

Congruence Method of generating pseudo-random numbers

\[ X_{n+1} = (aX_n + b) \mod T \]

\( a, b \) should be relatively prime

Example: \( a = \frac{3}{2}, b = \pi, T = 1 \)

(\( R \)) Seed \( X_0 = 1 \) \( \quad \text{NOTE: CAN GET FROM RANDOM \# TABLE} \)

\[
\begin{align*}
X_1 &= 0.1415926 \\
X_2 &= 0.641592654 \\
X_3 &= 0.103981635 \\
X_4 &= 0.297565
\end{align*}
\]

Can Show:

\[
\rho_s = E[X_nX_{n+s}] = 1 - \delta \frac{bs(T - bs)}{T} + \epsilon
\]

\( a_s = a^s \pmod T \)

\( b_s = (\sum_{n=b}^{s-1} a^n) b \pmod T \)

\(|\epsilon| < a_s / T \)

HP: \( X_{n+1} = \text{Fr}(X_n \pi)^5 \)
where \( p \sum_{n=0}^{\infty} p^n = \sum_{n=0}^{\infty} \left[ \frac{I}{\lambda} \right]^n = \frac{1}{1 - \frac{I}{\lambda}} \) for the mean of \( I \).

Given a random telegraph signal from a Poisson process, form a random process.

**Example:**

\[
E[I] = \frac{1}{2} e^{-I_{\text{ns}}} + \frac{1}{2} e - I_{\text{ns}} \delta(x - I_{\text{ns}})
\]

where \( I_{\text{ns}} \) is Poisson distributed.
<table>
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<tr>
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<th>Author</th>
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<tr>
<td></td>
<td>Thomas</td>
<td>&quot;An Intro to Statistical Communication Theory&quot; (Wiley, 1969)</td>
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<tr>
<td></td>
<td>Larson</td>
<td>&quot;Intro. to Probability Theory and Statistical Inference&quot; (Wiley)</td>
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<tr>
<td></td>
<td>Papoulis</td>
<td>&quot;The Fourier Integral and Its Applications&quot; (McGraw Hill)</td>
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EE 505

**INTRODUCTION TO PROBABILITY & RANDOM PROCESSES**

*Text: Papoulis Prob, Random Variables & Stochastic Processes*

**Grading:**
- **Homework:** 10% 15%
- **MidTerms:** 25% 40%
- **Final:** 40% 45%

**Bob Marks**

EE 404
3-6990

**Amir**

**Office Hrs:**
- 8-8:30 T. Th
- 10:30-noon W Th
1. Classic Definition

\[ P(A) = \frac{n_A}{n} \]

**Review of Probability**

**Definitions:**

1. **Relative Frequency (Empirical)**

\[ P(A) = \lim_{{n \to \infty}} \frac{n_A}{n} \]

**Ex:** \( A = \) Pencil crosses line

---

Ex: Roll 20 dice

\( A = \) Event that sum = 30

---

Ex:

\[ P_{\text{[IN CIRCLE]}} = \frac{\pi}{4} = \frac{\# \text{ IN}}{\# \text{ TOTAL}} \]

---

Monte Carlo Sim
M.S.E. (Mean Square Estimation) of a r.v. by a constant:

\[ \hat{Y} = a \]

Minimize Estimate's Uncertainty:

\[
E[(Y - a)^2] = \int_{-\infty}^{\infty} (y - a)^2 f_Y(y) \, dy
\]

\[
= \frac{1}{\sigma^2} \left[ y^2 - 2ay + a^2 \right] - \frac{1}{\sigma^2} \left[ y^2 - 2a\bar{Y} + a^2 \right]
\]

Can't change moments. Let

\[ a = \bar{Y} = \text{mean} \]

\[
\frac{1}{\sigma^2} \left( y^2 - 2Y + 2a \right) = 0 \Rightarrow a = \bar{Y}
\]
Nonlinear:
Mean-square estimation

From a r.v. \( X \), we wish to estimate the r.v. \( \hat{X} \), by
\[
\hat{X} = g(\bar{X}).
\]

Question: How to choose \( g \)?

Answer: Minimize m.s.e (mean square estimation):
\[
E \left[ (\hat{X} - g(\bar{X}))^2 \right]
= \int_{\mathbb{R}} \int_{\mathbb{R}} (y - g(x))^2 f_{\bar{X}, x}(x,y) \, dx \, dy.
\]

Note: minimizing square of "distance" 'twixt them.

Note:
\[
f_{\bar{X}, x}(x,y) = f_{x}(y/x) f_{\bar{X}}(x)
\]

\( \Rightarrow \)
\[
f_{\bar{X}, x}(x,y) = f_{y|x}(y/x) f_{\bar{X}}(x)
\]

Thus:
\[
E \left[ (\hat{X} - g(\bar{X}))^2 \right] = \int_{\mathbb{R}} f_{\bar{X}}(x) \int_{\mathbb{R}} (y - g(x))^2 f_{y|x}(y/x) \, dx \, dy
\]

\[
\text{Minimize } g \text{ at } x
\]

\[
\int_{\mathbb{R}} (y - g(x))^2 f_{y|x}(y/x) \, dy \leq \text{Second moment of } f_{y|x}(y/x) \text{ about the point } y = g(x) = \text{const}
\]

\[
= \int_{\mathbb{R}} y^2 f_{\bar{X}}(y+g(x)/x) \, dy
\]

\[
f_{\bar{X}}(y/x)
\]

\[
\text{Second moment}
\]

\[
g(x)
\]
Recall:

$$\sum (y - a)^2 f(y) \, dy = \frac{1}{2} \int (y - a)^2 \, f(y) \, dy$$

where $a = \bar{y}$, the second moment of $y$ about $a$.

Here:

$$E(x|\bar{y}) = \frac{\int x f(x|\bar{y}) \, dx}{\int f(x|\bar{y}) \, dx}$$

Thus, best m. s. e. is:

$$\sigma^2 = E[(y - \bar{y})^2]$$

$$s^2 = \frac{1}{n-1} \sum (y_i - \bar{y})^2$$

If $x$ and $y$ are ind.:

$$E[x] = E[\bar{y}]$$

$$E[x^2] = E[E[y|x]]$$

$$\sigma_x^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$$
\[
\mathbb{E}[X] = \int_0^\infty \int_0^\infty \frac{xe^{-x} \mu(x) \mu(y)}{\int_0^\infty \int_0^\infty (xe^{-x} \mu(x) \mu(y)) \, dx \, dy} \, dx \, dy
\]

\[
\frac{1}{X} = \frac{xe^{-x}}{\int_0^\infty xe^{-x} \mu(y) \, dy} \, dx
\]

\[
P_X = \left[ \frac{\partial}{\partial y} \right]_{y=0} \left[ \frac{\partial}{\partial x} \right]_{x=0} \mathbb{E}[X] = \left[ \frac{\partial}{\partial x} \right]_{x=0} \left[ \frac{\partial}{\partial y} \right]_{y=0} \mathbb{E}[X]
\]
Linear Mean-Square Estimation

assume $g(x) = ax + b$

Not as good $\rightarrow$ generally more tractable

$\Rightarrow$ Minimize $E \left[ (Y - (ax + b))^2 \right] = e$

$e = \iint (y - ax - b)^2 f(x,y) \, dx \, dy$

optimum:

$a = \frac{\rho \sigma_x}{\sigma_y}$

$b = \bar{y} - a \bar{x}$

$r = \frac{\bar{x} \bar{y}}{\sigma_x \sigma_y}$

gives

$e_m = \sigma_y^2 (1 - r^2)$

Proof: for given $a$, choose

$\Rightarrow$ obvious choice

Then:

$E \left[ (Y - ax - b)^2 \right]$

$= E \left[ (E - \bar{y}) - a (x - \bar{x}) \right]^2$

$= \sigma_y^2 - 2r \sigma_x \sigma_y \rho \sigma_y^2 q + \sigma_x^2 a^2$

Take $\frac{d}{da} = 0$

gives

$a = \frac{\rho \sigma^2}{\sigma_x}$

Note: If $X$ and $\bar{X}$ are zero mean, prob. becomes:

Minimize

$E \left[ (E - ax^3)^2 \right]$

gives

$e_m = \sigma_y^2 (1 - r^2)$
for $E[Y]$ zero mean

Proof: From previous result:

\[ E[(Y - \mu Y)^2] = \min \text{ for } a = \frac{\sigma_Y}{\sigma_X} \]

Thus

\[ E[(Y - \frac{\mu Y}{\sigma_X})^2] = 0 \]

\[ = \sigma_Y^2 - \frac{\sigma_Y}{\sigma_X} \frac{\sigma_X}{\sigma_X^2} \]

Q.E.D.

Note:

\[ E[(Y - \mu Y) (Y - \frac{\mu Y}{\sigma_X})] \]

\[ = \sigma_Y^2 - \frac{\sigma_Y}{\sigma_X} \frac{\sigma_X}{\sigma_X^2} \]

\[ = \sigma_Y^2 - \frac{\sigma_Y}{\sigma_X} \frac{\sigma_X}{\sigma_X^2} = (1 - r^2) \sigma_Y^2 \]

Alternate proof:

Note:

\[ E[(Y - \mu Y) (Y - \frac{\mu Y}{\sigma_X})] \]

\[ = \sigma_Y^2 - \frac{\sigma_Y}{\sigma_X} \frac{\sigma_X}{\sigma_X^2} \]

\[ = \sigma_Y^2 - \frac{\sigma_Y}{\sigma_X} \frac{\sigma_X}{\sigma_X^2} \]

\[ = (1 - r^2) \sigma_Y^2 \]
Two R.V. are "orthogonal" if
\[ xy = E[XY] = 0 \]

Suff cond: \( x \neq y \) are ind and one is zero mean

**ORTHOGONALITY PRINCIPLE:** The constant \( \theta \) that minimizes:

\[ e = E\left[ (Y - \theta X)^2 \right] \]

is such that \( Y - \theta X \) is orthogonal to \( X \). i.e.

\[ E\left[ (Y - \theta X)X \right] = 0 \]

Then:

\[ e_m = E\left[ (Y - \theta X)^2 \right] \]

**Hilbert space inter:**

\[ \text{to} \]

\[ \text{Much} \]

\[ \wedge \]
8. Sequences of Random Variables

\[ F(x_1, \ldots, x_n) = P[\sum_{i=1}^{n} x_i \leq x_1, \ldots, \sum_{i=1}^{n} x_i \leq x_n] \]

\[ f(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F(x_1, \ldots, x_n) \]

\[ f(x_1) = f_1(x_1) \cdot \prod_{i=2}^{n} f_i(x_i) \]
\begin{align*}
(x_1, \ldots, x_n) \\
\text{Uncorrelated:} \quad &E[x_i x_j] = E[x_i] E[x_j] \quad ; i \neq j \\
\text{Orthogonal:} \quad &E[x_i x_j] = \delta_{ij} \overline{x_i^2} \\
\text{Uncor. } \& \text{Orth. r.v.'s are invariant under linear combination, } \mu \quad &a_1 x_1 + \ldots + a_n x_n \text{ is uncor (orth)} \\
\text{For uncorrel:} & \quad \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2 \\
&= E\left[\left(\sum_{i=1}^{N} x_i^2 - N \overline{x}^2\right) - E[\sum_{i=1}^{N} x_i]\right]^2 \\
&= E\left[\sum_{i=1}^{N} (x_i - \overline{x})^2 - \overline{x}^2\right] \\
&= E\left[\sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} \overline{x}_n x_n - \overline{x}_n^2\right] \\
&= E\left[\sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} \overline{x}_n x_n\right] \\
&= E\left[\sum_{i=1}^{N} x_i^2\right] \\
&= E\left[\sum_{i=1}^{N} \overline{x}_n x_n\right] \\
&= E\left[\sum_{i=1}^{N} \overline{x}_n^2\right] \\
&= E\left[\sum_{i=1}^{N} \overline{x}_n^2\right] \\
&= \sigma^2_x + \sigma^2_x + \ldots
\end{align*}
IF orthogonal:
\[ E \left[ \left( \sum_{n=1}^{N} X_n \right)^2 \right] = \sum E \left[ X_n^2 \right] \]

MEAN SQUARE ESTIMATION.
\[ x_0, x_1, \ldots, x_N \]
wish to estimate \( x_0 \) in terms of \( x_1, \ldots, x_N \)
\[ \hat{x}_0 = g(x, \ldots, x_N) \]

MINIMIZE M.S. Error:
\[ E \left[ \left( \hat{x}_0 - g(x, \ldots, x_N) \right)^2 \right] \]

Generalization of previous result:

Choose:
\[ g(x, \ldots, x_N) = E \left[ x_0 \mid x_1, \ldots, x_N \right] \]

MUST know density of
\[ x_0, \ldots, x_N \]
Sample Means & Estimates from Char. Func.

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \]

Assume: \( x_i \) are independent.

\[ E[x_i] = \mu \]

\[ \text{var} x_i = \sigma^2 \]

\[ E[x_i^2] = \sigma^2 + \mu^2 \]

\[ E[\bar{x}] = \mu \]

\[ \text{var} \bar{x} = \frac{\sigma^2}{n} \]

\[ \text{alt. will show:} \]

\[ E[\bar{x}] = \mu \]

\[ \text{var} \bar{x} = \frac{\sigma^2}{n} \]
**Proof**

\[ E(x) = \frac{1}{n} \sum_{i=1}^{n} x_i \]

\[ = \frac{1}{n} \prod_{i=1}^{n} x_i \]

\[ = \frac{1}{n} \prod_{i=1}^{n} \xi \]

\[ = \phi_x^n(\frac{\omega}{n}) \]

\[ \Rightarrow \phi_x(\omega) = \phi_x^n(\frac{\omega}{n}) \]

\[ \frac{d}{d\omega} \phi_x = n \cdot \frac{1}{n} \phi_x^{n-1}(\frac{\omega}{n}) \phi'_x(\frac{\omega}{n}) \]

\[ @ \omega = 0 \Rightarrow \phi_x = 1 \times \phi_x \]

\[ \frac{d^2}{d\omega^2} \phi_x = \frac{n-1}{n} \phi_x^{n-2}(\frac{\omega}{n}) \left[ \phi_x'(\frac{\omega}{n}) \right]^2 + \frac{1}{n} \phi_x^{n-1}(\frac{\omega}{n}) \phi''(\frac{\omega}{n}) \]

\[ @ \omega = 0 \Rightarrow E(x^2) = \frac{n-1}{n} \sigma^2 + \frac{1}{n} (\sigma^2 - \xi^2) \]

\[ \sigma_x^2 = (1 - \frac{1}{n}) \xi^2 + \frac{1}{n} (\sigma^2 + \xi^2) - \xi^2 \]

\[ = \frac{1}{n} \xi^2 + \frac{1}{n} \sigma^2 - \frac{\xi^2}{n} \]

\[ = \frac{\sigma^2}{n} \]
\[ \lim_{n \to \infty} \frac{\sigma^2}{n} = 0 \quad \text{IN LIMIT:} \]

\[ \frac{2 \pi e^{(x - 2)^2}}{2!} \]

**Markoff's Proof of "Law of Large #s"**

(p. 265)

**Elaborate**

**Central Limit Theorem**

\[ \frac{\sigma^2 + \sigma^2 + \ldots + \sigma^2}{n} \to 0 \]

\[ \exists \alpha > 0 \]

\[ \exists \int_{-\infty}^{\infty} x^2 f(x) \, dx < C \]
Sample mean & variance; \( x_i \sim f_X(x_i), \overline{x} = \frac{1}{N} \sum x_i \), \( \bar{X}^2 = \frac{1}{N} \sum (x_i - \overline{x})^2 \)

Ind. \( \sim \) uncorr:

\[ E\left[ (x_i - \overline{x})(x_k - \overline{x}) \right] = \sigma^2 \]

To prove:

1. \( E[\overline{x}] = \sigma \)
2. \( \text{var} \overline{x} = \frac{\sigma^2}{N} \)
3. \( E[\bar{X}^2] = \frac{n-1}{n} \sigma^2 \)

Proof:

1. \( E[\overline{x}] = \frac{\sum x_i}{n} = \sigma \)

Since uncorrelated:

2. \( \sigma^2_x = \frac{1}{N^2} (\sigma^2 + \ldots + \sigma^2) = \frac{\sigma^2}{N} \)

3. Lastly:

   \[ E\left[ (x_i - \overline{x})(x - \overline{x}) \right] = E\left[ (x_i - \overline{x}) \right] \]

   \[ = E\left[ (x_i - \overline{x}) \right] \]

   \[ = \sigma^2 \]

Thus:

\[ E\left[ (x_i - \overline{x})^2 \right] = E\left[ \left( (x_i - \overline{x}) - (\overline{x} - \overline{x}) \right)^2 \right] \]

\[ = \sigma^2 - 2 \frac{\sigma^2}{N} + \frac{\sigma^2}{N} \]

\[ = \frac{N-1}{N} \sigma^2 \]

Thus:

\[ E[\bar{X}^2] = E\left[ \frac{(x_i - \overline{x})^2 + \ldots + (x_i - \overline{x})^2}{N} \right] \]

\[ = \frac{(N-1)\sigma^2}{N} \cdot \frac{N}{N} \]

Q.E.D.
NORMAL RANDOM VARIABLES

\( N \) th order zero mean, equal variance \( \sigma^2 \)

\[ f_{x_1, \ldots, x_n}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} \sigma^n} e^{-\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{2\sigma^2}} \]

Let:

\[ \bar{x} = \frac{x_1 + \ldots + x_n}{n} \]

\[ s^2 = \frac{(x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n} \]

\[ X = \sqrt{x_1^2 + \ldots + x_n^2} \]

\[ y = x_1^2 + \ldots + x_n^2 \]

Densities for each:

\[ f_x \sim \text{normal } (0, \text{ var } = \frac{\sigma^2}{N}) \]

\[ = \frac{1}{\sqrt{2\pi} \sigma^2 N} e^{-\frac{x^2}{2\sigma^2}} \]

For \( X \)

\[ X \leq \sqrt{x_1^2 + \ldots + x_n^2} \leq X + dX \]

Hypershell

dv = \( \chi^{n-1} \) dx

In shell; mass is:

\[ \frac{1}{(2\pi)^{n/2} \sigma^n} e^{-\frac{x^2}{2\sigma^2}} dx \]

\[ \Rightarrow f_X(x) = \frac{\text{CONST}}{(2\pi)^{n/2} \sigma^n} x^{n-1} e^{-\frac{x^2}{2\sigma^2}} dx \]

can find CONST via unit area:

\[ f_X(x) = \frac{\pi}{2}^{n/2} \sigma^n \Gamma\left(\frac{n}{2}\right) x^{n-1} e^{-\frac{x^2}{2\sigma^2}} \mu(x) \]
For $y = x^2$:

$$f_y(y) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right)$$

Special case of Gamma distribution specified by: $\alpha = 0.5 - n$ degrees of freedom
For $n = 2$

$$V = \frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} \right)^2 + \frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^{n} x_i + \frac{1}{n} \right)^2$$

but $z_1 = \frac{x_i - \bar{x}}{\sigma}$

The population has $n = 3$.
Alternate treatment: \( \overline{X}_n \sim \frac{N}{\sigma^2}, \overline{X}_n \) normal \( \overline{X}_n \sim \frac{N}{\sigma^2} \)

\[
\bar{X} = \frac{1}{N} \sum_{n=1}^{N} (X_n - \bar{X})
\]

\[
\bar{X} = \frac{1}{N} \sum_{n=1}^{N} (X_n - \bar{X})^{-1}
\]

Showed in H.W:

\( \overline{V} \neq \overline{X} \) are ind.

Proof that \( \frac{N\overline{S}^2}{\sigma^2} \) is \( \chi^2 \).

We know:

\[
\frac{1}{N} \sum_{n=1}^{N} (X_n - \overline{X})^2 \sim \chi^2_n \quad (\sigma^2 = 1)
\]

Now

\[
\sum_{n=1}^{N} (X_n - \mu)^2 = \sum_{n=1}^{N} (X_n - \overline{X} + \overline{X} - \mu)^2
\]

\[
= \sum_{n=1}^{N} (X_n - \overline{X})^2 + N(\overline{X} - \mu)^2
\]

(since \( z(\overline{X} - \mu) \sum_{n=1}^{N} (X_n - \overline{X}) = 0 \))

Thus:

\[
\frac{N}{\sigma^2} \sum_{n=1}^{N} (X_n - \mu)^2 = \frac{N}{\sigma^2} \sum_{n=1}^{N} (X_n - \overline{X})^2
\]

\[
= \frac{N(\overline{X} - \mu)^2}{\sigma^2} + \frac{N\overline{V}}{\sigma^2}
\]
\[
\frac{\sum_{i=1}^{N} (x_i - \mu)^2}{\sigma^2} \rightarrow \chi^2_n \quad \text{(Also ind (showed in))}
\]

Theorem: \( \sum \chi^2 \sim \chi^2_n \quad \rightarrow \quad \chi^2_x \sim \chi^2_m \quad z = \chi^2_x \quad \sigma^2 = 1 \)

\( I + Z = \chi^2_{n+m} \)

(Prove via char. functions)

Hence \( \frac{N \bar{v}}{\sigma^2} \sim \chi^2_{n-1} \)
Recall \( \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 \) = \( \frac{1}{n} \sum_{i=1}^{n} z_i^2 \)

\( z = \frac{x - \bar{x}}{\sigma} \) zero mean, unit variance gaussian r.v.

Why \( n-1 \) \( z_i \) not \( n \) \( z_i \)'s (since there are \( n \) \( x_i \)'s)

\( \bar{v} = \bar{c}_1^2 + \ldots + \bar{c}_n^2 \)

But \( c_i^2 \)'s not ind.

\( \sum_{i=1}^{n} c_i^2 = 0 \)

Let since

\( c_n = -(c_1 + \ldots + c_{n-1}) \)

Substituting \( c_n \) complete square results in \( \bar{v} \) expression

\( z \)'s zero mean, normal with zero mean \& equal variance

\( \Rightarrow \bar{v} \sim \chi^2 \) special case of gamma

\( \Rightarrow P(\frac{v}{\sigma^2}) \sim \chi^2 \)

\( E(v) = \frac{n-1}{n} \sigma^2 \)

\( f_v(v) = \frac{v^{n-1/2} \Gamma(n/2)}{2^{n/2} \Gamma(n/2)} e^{-v/2\sigma^2} \)

\( \Gamma(n/2) \Gamma(n/2-1) \)
**LINEAR MEAN SQUARE ESTIMATION. MINIMIZE**

\[ e = E \left[ (x_0 - (a_1 x_1 + a_2 x_2 + \ldots + a_n x_n))^2 \right] \]

Define:

\[ R_{ij} = x_i x_j = E [x_i x_j] \]

If \( x_i = 0 \), \( R_{ij} = \text{covariance of } x_i \text{ and } x_j \)

\[ \text{Wish to have error:} \]

\[ x_0 - (a_1 x_1 + \ldots + a_n x_n) \]

is orthogonal to \((x_1, \ldots, x_n)\). That is:

\[ * E \left[ (x_0 - (a_1 x_1 + \ldots + a_n x_n)) x_i \right] = 0; \ i = 1, 2, \ldots, n \]

**Proof:**

To minimize, set

\[ \frac{\delta e}{\delta a_i} = 0 = \frac{\delta E \left[ (x_0 - (a_1 x_1 + \ldots + a_n x_n))^2 \right]}{\delta a_i} \]

\[ = -2E \left[ x_0 - (a_1 x_1 + \ldots + a_n x_n) \right] x_i = 0 \]

orthogonality principle follows.

**What is minimal ERROR?**

Expanding \( E \) using orthogonality principle:

\[ E_m = E \left[ (x_0 - (a_1 x_1 + \ldots + a_n x_n))^2 \right] = E \left[ (x_0 - (a_1 x_1 + \ldots + a_n x_n)) \right]^2 \]

\[ = R_{00} - (a_1 R_{11} + a_2 R_{22} + \ldots + a_n R_{nn}) \]

Solving for \( a_i \)’s:

\[ R_{01} = a_1 R_{11} + a_2 R_{12} + \ldots + a_n R_{1n} \]

\[ R_{22} = a_1 R_{22} + a_2 R_{22} + \ldots + a_n R_{2n} \]

\[ R_{nn} = a_1 R_{nn} + a_2 R_{n2} + \ldots + a_n R_{nn} \]

\[ \text{or} \quad \vec{R}_0 = \hat{a} = \vec{R} \hat{a} \]
Central Limit Theorem

Let $X_n, n = 1, 2, 3, \ldots, N$

\[ E[X_n] = \bar{x}_n \]
\[ \text{var } X_n = \sigma_n^2 \]

Let:

\[ \bar{Y}_N = \frac{\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \bar{x}_n}{\sqrt{\sum_{n=1}^{N} \sigma_n^2}} \]

Then

Note:

\[ E[\bar{Y}_N] = 0 \]
\[ \text{var } \bar{Y}_N = 1 \quad \text{ (prove)} \]

Then, as $N \to \infty$, under rather weak conditions:

\[ \bar{Y}_N \sim \text{normal } (0, 1) \]

Under

Sufficient:

\[ E \left[ \sum_{n=1}^{N} \bar{x}_n \right] = 0 \]
\[ \sigma_n^2 > m > 0 \]
\[ E \left[ |X_n - \bar{x}_n|^{3/2} \right] < M \]

Assures 1 term don't dominate
Special Case:

Identically distributed: \( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (X_n - \bar{X}) \)

Let \( Z_n = \frac{X_n - \bar{X}}{\sigma} \)

Then: \( E[Z_n] = 0 \), \( \text{var } Z_n = 1 \)

Proof:

\[ E[Z_n] = E \left[ \frac{X_n - \bar{X}}{\sigma} \right] \]

and:

\[ Y_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Z_n \]

\[ \Phi_{Y_N}(\omega) = E \left[ e^{i\omega Y_N} \right] \]

\[ = E \left[ e^{i\omega \frac{1}{\sqrt{N}} \sum_{n=1}^{N} Z_n} \right] \]

\[ = \Phi_{Z} \left( \frac{\omega}{\sqrt{N}} \right) \]

\[ \Phi_{Z}(\omega) = \sum_{n=0}^{\infty} \frac{(i\omega)^n}{n!} \bar{z}^n \]

\[ = 1 + j\omega \bar{z} \frac{\bar{z}}{1!} + (j\omega)^2 \frac{\bar{z}^2}{2!} + \frac{(j\omega)^3 \bar{z}^3}{3!} + \text{h.o.t.} \]

\[ \bar{z} = 0, \bar{z}^2 = \text{var } z = 1 \]

\[ = 1 + 0 + (j\omega) \frac{1}{2!} \frac{\bar{z}^2}{2!} + \frac{(j\omega)^3 \bar{z}^3}{3!} + \text{h.o.t.} \]

\[ \Phi_{Z} \left( \frac{\omega}{\sqrt{N}} \right) = 1 + \frac{(i\omega)^2 \bar{z}^2}{N^2 2!} + \frac{(j\omega)^3 \bar{z}^3}{N^3 3!} + \text{h.o.t.} \]

Large \( N \):

\[ \Phi_{Z} \left( \frac{\omega}{\sqrt{N}} \right) \approx 1 - \frac{\omega^2}{2N} \approx e^{-\frac{\omega^2}{2N}} \]

\[ \Rightarrow \Phi_{Y_N}(\omega) \rightarrow \text{char func of normal distribution} \]
A CLT Example from IRS

48

20 number added rounded to nearest dollars

\[ S = \sum_{n=1}^{N} a_n, \quad \bar{S} = \frac{1}{N} \sum_{n=1}^{N} a_n \]

\[ E = S - \bar{S} = \sum_{n=1}^{N} \epsilon_n \]

\[ \epsilon \sim \mathcal{N}(0, \frac{1}{12}) \]

By CLT, \( E \sim N(0, \frac{N}{12}) \)

For \( N = 120 \), \( E \sim N(0, 10) \)

\[ \Pr[-D \leq E \leq D] = \Pr[\text{off by no more than } \# D] \]

\[ = \int_{-D}^{D} \frac{1}{\sqrt{2\pi} \sqrt{N/12}} e^{-\frac{x^2}{2(N/12)}} \, dx \]

\[ y = \frac{x}{\sqrt{N/12}} \quad d = \frac{\sqrt{N/12}}{\sqrt{2\pi}} \]

\[ = \int_{-D}^{D} \frac{1}{\sqrt{2\pi} \sqrt{12}} e^{-\frac{x^2}{2}} \, dx \]

\[ = 2 \text{erf} D \sqrt{\frac{N}{12}} \]

\[ N = 120 \Rightarrow 2 \text{erf} 2 \sqrt{10} \]

\[ 0.95 \quad -2 \text{erf} 2 \]

\[ 0.68 \]

\[ 0.68 \leq \frac{1}{2} \]

\[ D = 5.04 \]

\[ 2 \text{erf} 1 = 0.68 \]
Joint Char Function:

\[ \Phi_{x_1 x_2}(\omega, \omega_2) = E \left[ e^{i(w. X + \omega_2 . X)} \right] \]

\[ \frac{\delta^k \delta^r \Phi(0,0)}{\delta \omega_1^k \delta \omega_2^r} = (i)^{k+r} M_{k,r} \]

(Prove)

If Ind
STOCHASTIC PROCESSES

$X(t)$

Random variable depends on time $t$.
Set $t$.
Define
Corresponding R.V. has pdf:

$$f(x; t) = \frac{e^{-\frac{x}{\theta}}}{\theta}$$

Ensemble (rel freq)
\( \mathbb{S}(\cdot) \) is related (maybe) to \( \mathbb{X}(\cdot) \).

Define:

\[
F(x, x_2; t, t_2) = P, \quad \mathbb{X}(x, x_2; t, t_2) \leq x_2
\]

\[
f(x_1, x_2; t, t_2) = \frac{2}{\mathcal{B}_x \mathcal{B}_{x_2}} F(x, x_2; t, t_2)
\]

\[
f_{\text{first order distr.}}(x, x_2; t, t_2) dx_2
\]

\[
f(x; t, t) = \int_{x_2} f_{\text{second order distr.}}(x, x_2; t, t_2) dx_2
\]
Moments:
\[ \hat{\mu}(t) = E(x(t)) = \int_{-\infty}^{\infty} x \cdot f(x; t) \, dx \]

Autocorrelation
\[ R(t_1, t_2) = E[x(t_1) \cdot x(t_2)] \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 \cdot x_2 \cdot f(x_1, x_2; t_1, t_2) \, dx_1 \, dx_2 \]

Auto covariance:
\[ C(t_1, t_2) = E[(x(t_1) - \mu(t_1))(x(t_2) - \mu(t_2))] \]
\[ = R(t_1, t_2) - \mu(t_1) \cdot \gamma(t_2) \]

Note:
\[ \sigma^2_x(t) = C(t, t) \]
\[ = R(t, t) - \mu^2(t) \]
\[ E_x \ X(t) = u(t - \tau) \]
\[ \tau = \begin{cases} 1 & t < 0 \\ 0 & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases} \]

\[ R(t_1, t_2) = E(X(t_1), X(t_2)) \]
\[ \text{assume } t_1, t_2 \geq 0 \Rightarrow R(t_1, t_2) = \int_0^t \int_0^t u(t_1 - \tau) u(t_2 - \tau) d\tau \]
\[ \text{assume } t_1 \leq t_2 \]

1. \( t_1 > 1, t_2 > 1 \Rightarrow R(t_1, t_2) = 0 \)
2. \( t_1 > 1, 0 < t_2 < 1 \Rightarrow R(t_1, t_2) = 1 - t_2 \)
3. \( 0 \leq t_1, t_2 \leq 1 \Rightarrow R(t_1, t_2) = 1 - t_1 \)
4. \( 0 \leq t_1, t_2 \leq 1 \Rightarrow R(t_1, t_2) = (1 - t_1)(1 - t_2) \)
(1) $t_1 > 1$ or $t_2 > 1 \Rightarrow R(t_1, t_2) = 0$
(2) $0 \leq t_1 < 1$, $t_2 < 0 \Rightarrow R(t_1, t_2) = (1-t_1)(1-t_2)$
(3) $t_1 < 0$, $0 < t_2 < 1 \Rightarrow R(t_1, t_2) = 1$
(4) $-1 < t_1 < 0$, $t_2 < 0 \Rightarrow R(t_1, t_2) = 1$
(5) $t_1 < 0$, $t_2 < 0$
EXAMPLE:

\[
E[x(t)] = E(r \cos(\omega t + \Phi)) = E(r_1 \cos(\omega t_1 + \Phi))
\]

\[
E[x(t_1)] E[x(t_2)] = E[r_1 \cos(\omega t_1 + \Phi)] E[r_1 \cos(\omega t_2 + \Phi)]
\]

\[
= \frac{1}{2} E[r_1^2] E[\cos(\omega t_1 + \Phi)] E[\cos(\omega t_2 + \Phi)]
\]

But

\[
E[\cos(\omega t_1 + \Phi)] E[\cos(\omega t_2 - 2\Phi)]
\]

\[
= \frac{1}{2} \int_{-\pi}^{\pi} \cos(\omega t_1 + \Phi) \cos(\omega t_2 - 2\Phi) d\Phi = 0
\]

Thus:

\[
R(t_1, t_2) = \frac{1}{2} E[r_1^2] \cos \omega (t_1 - t_2)
\]
**Discrete Time Processes**

\[ x[n] \]

\[ \begin{array}{cccc}
    0 & 1 & 2 & 3 \\
    \uparrow & \uparrow & \uparrow \\
    n & \text{mean} & \text{autocorrelation} & \text{autocovariance}
\end{array} \]

\[ \mu[n] = E[x[n]] \]

\[ R[n_1, n_2] = E[x[n_1] x^*[n_2]] \]

\[ C[n, n_2] = R[n, n_2] - \mu[n] \mu^*[n_2] \]

**White noise:**

\[ C[n, n_2] = q[n] \delta[n_1 - n_2] \]

\[ q[n] = E[x^2[n]] \]

\( x[n] \) is wss if

\[ E[x[n + m] x^*[n]] = R[m] \]

**Sampling**

\[ x[n] = x(nT) \]

\[ \mu[n] = \mu(nT) \]

\[ R[n, n_2] = R(n, T, n_2 T) \]

If stationary:

\[ R[m] = R(mT) \]
\( E_x \ x(t) = p + q t \)
\[
p + f_p \rightarrow q + f_q
\]
\[ z(t) = E[x(t)] = E[p + q t] \]
\[
= E(p) + E(q) t
\]
\[ R(t, t_2) = E \left[ (p + q t_1) (p + q t_2) \right] \]
\[
= E(p^2) + E(p q) (t_1 + t_2) + E(q^2) t_1 t_2
\]
\[ C(t_1, t_2) = \sigma_p^2 + t_1 t_2 \sigma_q^2 \]
Special Processes:

**Poisson**

**Place n points on interval** \((0, T)\)

**Let** \(t_2 - t_1 = t_0\)

**What is** \(Pr[k \leq t_0]\)?

\[
Pr[k \leq t_0] = \binom{n}{k} p^k q^{n-k} \quad p = \frac{t_0}{t}
\]

If \(n \gg 1\), \(p \approx = \frac{t_0}{t} \ll 1\)

\[
Pr[k \leq t_0] \approx e^{-n \frac{t_0}{t}} \left( \frac{nt_0}{t} \right)^k \frac{1}{k!}
\]

Define **Poisson process** as follows:

Let \(X(0) = 0\)

\[X(t) = \text{# points on interval} \ (0, t)\]

**Equivalently:**

\[X(t_0) - X(t_0) = \text{# points} \text{ in } t_0 \}

\[t_0 \geq t_0\]

\[\frac{X(t)}{t}

\[\text{have } \frac{n}{\lambda}

\text{Expected
First order

STATISTICS:

\[ t_a > t_b \]

\[ \Pr \left[ X(t_a) - X(t_b) = k \right] = e^{-\lambda (t_a - t_b)} \frac{\lambda (t_a - t_b)^k}{k!} \]

Recall:

\[ f_x(x) = e^{-\frac{x}{a}} \sum_{k=0}^{\infty} \frac{a^k}{k!} \delta(x-k) ; a = \lambda(t_a - t_b) \]

\[ E(x) = a \]

\[ \text{Var}(x) = \frac{a^2}{2} \]

\[ \Rightarrow E(x^2) = a^2 - a \]

\[ \therefore E[X(t_a) - X(t_b)] = \lambda(t_a - t_b) \]

\[ E[(X(t_a) - X(t_b))^2] = \lambda^2(t_a - t_b)^2 + \lambda(t_a - t_b) \]
Second Order

For Poisson Process:

\[ R(t, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t, t_2) \]

Proof: For \( t_1 < t_2 \)

\( X(t_1) \) is ind. of \( X(t_2) - X(t_1) \) if \( t_1 < t_2 \)

(just like Weiner)

\[
E[X(t_1) [X(t_2) - X(t_1)]]
= E[X(t_1)] E[X(t_2) - X(t_1)]
= \lambda t_1 \cdot \lambda (t_2 - t_1)
\]

Since:

\[
X(t_1) X(t_2) = X(t_1) [X(t_1) + X(t_2) - X(t_1)]
\]

We have

\[
R(t, t_2) = E[X(t_1) X(t_2)]
= \overline{X^2(t_1)} + \lambda^2 t_1 (t_2 - t_1)
\]

\( \overline{X^2(t_1)} = \text{second moment of Poisson RV} \)

\[
= \lambda t_1 + \lambda^2 t_1^2
\]

\[ R(t_1, t_2) = \lambda t_1 + \lambda^2 t_1^2 + \lambda t_1 (t_2 - t_1); t_1 < t_2
\]

\[ = \lambda t_1 + \lambda^2 t_1 t_2
\]

for \( t_1 > t_2 \), switch. Q.E.D.
STATIONARY IN WIDE SENSE
(WEAKLY STATIONARY)

\[ 1 = E[X(t)] = \mu \]
\[ E[X(t+\tau)X(-\tau)] = R(\tau) \]
STATIONARY PROCESSES

STRICT SENSE STATIONARITY

\[ X(t) \] IS STRICTLY STATIONARY IF IT HAS THE SAME STATISTICS AS \[ X(t+\epsilon) \]

\[ X(t) \neq X(t) \] HAVE ARE JOINTLY STATIONARY IF SAME STATISTICS AS \[ X(t+\epsilon), X(t+\epsilon) \]

\[ X(t) \neq X(t) \] MIGHT BE INDIVIDUALLY BUT NOT JOINTLY STATIONARY

FOR SR. STATIONARY R.V.

\[ E[X(t)] = E[X(t+\epsilon)] \neq \epsilon \]

\[ \Rightarrow \mu(t) = \mu = \text{CONSTANT} \]
Since
\[ f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 + \epsilon, t_2 + \epsilon) \]
\[ R(t_1, t_2) = E[x(t_1) \cdot x^*(t_2)] = E[x(t_1 + \epsilon) \cdot x^*(t_2 + \epsilon)] \]
\[ = R(t_1 - t_2) = R(\tau) \]
\[ R(\tau) = E[x(t + \tau) \cdot x^*(t)] = R(-\tau) \]

Cross correlation:
\[ R_{xy}(\tau) = E[x(t + \tau) \cdot y^*(t)] \]
\[ X(t) \text{ is wss} \]
\[ S = \int_{-T}^{T} X(t) \, dt \]
\[ E[S] = 2T \mu ; \quad \mu = E[X] \]
\[ \sigma_s^2 = E[(S - \mu)^2] \]
\[ = \int_{-T}^{T} \int_{-T}^{T} E[X(t_1)X(t_2)] \, dt_1 \, dt_2 \]
\[ = \int_{-T}^{T} \int_{-T}^{T} R(t_1, t_2) \, dt_1 \, dt_2 \]
\[ = \int_{-T}^{T} \int_{-T}^{T} R(t_2 - t_1) \, dt_1 \, dt_2 \]
\[ \tau = t_2 - t_1 \Rightarrow t_1 = t_2 - \tau \]
\[ E[S^2] = \int_{-T}^{T} \int_{-\infty}^{\infty} \text{rect} \left( \frac{t_2 - \tau}{2T} \right) R(\tau) \, d\tau \, dt_2 \]
\[ = \int_{-\infty}^{\infty} R(\tau) \int_{T=\infty}^{\infty} \text{rect} \left( \frac{t_2 - \tau}{2T} \right) \text{rect} \left( \frac{t_2 - \tau}{2T} \right) \, dt_2 \]
\[ = \int_{T=-2T}^{2T} R(\tau) \left[ 2T - \left| \tau + \frac{1}{2} \right| \right] \, d\tau \]

Similarly,
\[ \sigma_s^2 = \int_{-2T}^{2T} (2T - \left| \tau + \frac{1}{2} \right|) C(\tau) \, d\tau \]
\[ E_x \]
\[ a_i \sim i = 1, 2, \ldots, N \quad \text{and} \quad \text{uncorrelated, } E[a_i] = 0, \quad E[a_i a_j] = 0 \]
\[ E[x(t)] = \sum_{n=1}^{N} a_i e^{j\omega n t} \] (Trig. polynomial)
\[ E[x^2(t)] = \sum_{n=1}^{N} E[a_i^2] \]
\[ R(\tau) = E[x(t+\tau)x^*(t)] \]
\[ = \sum_{n=1}^{N} \sum_{m=1}^{N} a_i^2 e^{j\omega m t} e^{-j\omega n t} \]
\[ = \sum_{n=1}^{N} \text{wide sense stationary?} \]

\[ \text{Ex.} \]
\[ a \perp b; \quad \text{uncorrelated, zero mean} \]
\[ E[a_i^2] = E[b_i^2] = 0; \quad \omega_i \]
\[ x(t) = \sum_{i=1}^{N} (a_i \cos \omega t + b_i \sin \omega t) \]
\[ \text{Let } c_i = a_i - j b_i \]
\[ y(t) = \sum_{i=1}^{N} c_i e^{j\omega n t} \]
\[ x(t) = \Re y(t) \]

Follows from prev. \( \text{Ex.} \)
\[ R(\tau) = \Re \sum_{n=1}^{N} a_i^2 e^{j\omega n t} \]
\[ = \sum_{n=1}^{N} a_i^2 \cos \omega n t \]
Consider ST. R. Processes

\[
\begin{align*}
    R_x(t) & = E[x(t) \cdot x(t)] = R_{xx}(t) = \text{auto-cor.} \\
    R_{xy}(t) & = E[x(t) \cdot y(t)] = R_{xy}(t) = \text{cross-cor.}
\end{align*}
\]
\[
E_{x, y} = \mathbb{E}_{x,y}[E_{x, y}(t)] = 0
\]

Then:

\[
R_m(t) = R_x(t)R_y(t)
\]

If \( R(t) = E_{x, y}(t) \), then:

\[
R(0) = \mathbb{E}_{x,y}[^{+}x(t)^{+}y(t)] \geq 0
\]

If \( R(t) = E_{x, y}(t) \), then:

\[
1 \leq \mathbb{E}_{x,y}[E_{x, y}(t)] = E_{x, y}[^{+}x(t)^{+}y(t)] \geq \mathbb{E}_{x,y}[E_{x, y}(t)] = E_{x, y}[E_{x, y}(t)]
\]

\[
R(0) \geq R(t)
\]

\[
R(0) \leq \mathbb{E}_{x,y}[E_{x, y}(t)] = E_{x, y}[E_{x, y}(t)]
\]

\[
| R(t) | \leq R(0) \Rightarrow R(0) = \max_{t}
\]

For real process:

\[
R(0) = 1
\]

\[
R(0) \neq R(t)
\]

\[
R(0) \neq R(t)
\]

\[
R(0) \neq R(t)
\]
Fourier Integrals of Stochastic Processes

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

Stochastic Processes

Inversion:

\[ X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]

Properties:

1. First order statistics:

   \[ E[X(\omega)] = \int_{-\infty}^{\infty} E[x(t)] e^{-j\omega t} dt \]

   \[ E[X(\omega)] = \overline{\mathbb{E}_x(t)} \]

   note: For stationary process, \( E[X(\omega)] = \infty \)

   \( \mathbb{N} \neq 0 \)

2. Second order statistics:

   \[ E[X(u)X^*(v)] = \Gamma(u, -v) \]

   where

   \[ \Gamma(u, v) = \int_{-\infty}^{\infty} R(t_1, t_2) e^{-j(ut_1 + vt_2)} dt_1 dt_2 \]

Proof:

\[ E[X(u)X^*(v)] = E \int_{t_1} x(t_1) e^{-jut_1} dt_1 \int_{t_2} x^*(t_2) e^{jvt_2} dt_2 \]

\[ = \int_{t_1} \int_{t_2} R(t_1, t_2) e^{-j(ut_1 + (-v)t_2)} dt_1 dt_2 \]
3. If \( x(t) \) is WSS with power spectrum \( S(\omega) \), then \( X(\omega) \) is white noise with average intensity \( 2\pi S(u) \):

\[
E[X(u)X^*(\nu)] = 2\pi S(u) \delta(u-\nu)
\]

Proof: If \( \not\text{WSS} \), \( R(t_1,t_2) = R(t_1-t_2) \)

\[
\Gamma(u,v) = \int_{-\infty}^{\infty} R(t_1-t_2) e^{-j(u t_1 + \nu t_2)} dt_1 dt_2
\]

\[
= \int_{-\infty}^{\infty} e^{-j(u+v)t_2} \int_{-\infty}^{\infty} R(\tau) e^{-j\tau t_1} d\tau dt_2
\]

\[
= S(u) \int_{-\infty}^{\infty} e^{-j(u+v)t_2} dt_2
\]

\[
= S(u) \ 2\pi S(v+u)
\]
4. If \( x(t) \) is white:
\[
R(t, t_2) = q(t_1) \delta(t_2 - t_1)
\]
then \( X(w) \) is WSS with autocorrelation:
\[
E[X(u)X^*(v)] = Q(u-v)
\]
where
\[
Q(w) = \int_{-\infty}^{\infty} q(t) e^{-j \omega t} dt
\]

Proof:
\[
\Gamma(u,v) = \iint q(t_1) \delta(t_1 - t_2) e^{-j (ut_1 + vt_2)} dt_1 dt_2
\]
\[
= \int q(t_1) e^{-j (u+v)t_2} dt_2
\]
\[
= Q(u+v)
\]

Note:
\[
E[|X(w)|^2] = Q(0) = \int_{-\infty}^{\infty} q(t) dt
\]

(look at 10:34 for final)
Power Spectrum

\[ S(\omega) = \text{Power Spectrum (Spectral Density)} \]
\[ = \int_{-\infty}^{\infty} e^{-j\omega \tau} R(\tau) d\tau \]

Since \( R(\tau) = R^*(-\tau) \) (Hermitian)
\( S(\omega) \) is Real Function \( \geq 0 \)

Inversion

\[ R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega \tau} d\omega \]

Note:

\[ R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = E\left\{ |X(t)|^2 \right\} \]

Equivalently:

\[ S(\omega) = \int_{-\infty}^{\infty} R(\tau) \cos \omega \tau d\tau \text{ if \( \Re \omega \)} \]
\[ R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \cos \omega \tau d\omega \]
LINEAR SYSTEMS:

\[
\begin{align*}
X(t) \rightarrow h(t) \rightarrow Y(t)
\end{align*}
\]

\[
Y(t) = X(t) * h(t)
\]

\[
= \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau
\]

**Assume \( X \) is STATIONARY:**

\[
\mathbb{E}[Y(t)] = \int_{-\infty}^{\infty} \mathbb{E}[X(\tau)] h(t-\tau) d\tau = \mathbb{E}[X] \int_{-\infty}^{\infty} h(\tau) d\tau = \mathbb{E}[X] H(0)
\]

**Auto Correlation:**

\[
R_y(\tau) = \mathbb{E}[Y(t) Y(t+\tau)]
\]

\[
= \mathbb{E}\left[ \int_{-\infty}^{\infty} X(\alpha) h(t-\alpha) d\alpha \right] \left[ \int_{-\infty}^{\infty} X(\beta) h(t+\beta) d\beta \right]
\]

\[
= \mathbb{E}\left[ \int_{-\infty}^{\infty} X(\alpha) X(\beta) d\alpha d\beta \right]
\]

**Consider:**

\[
Y(t) \times X(t-\tau) = \int \overline{X}(t-\alpha) \overline{X}^*(t-\tau) h(\alpha) d\alpha
\]

\[
R_{xy}(\tau) = \mathbb{E}[Y(t) \times X(t-\tau)]
\]

\[
= \int R_x(\tau-\alpha) h(\alpha) d\alpha
\]

\[
R_y(\tau) = R_x(\tau) * h(\tau)
\]
\[ Y(t + \tau) \ast h(t) = \int_{-\infty}^{\infty} R_{xy}(\tau - \alpha) \ast h(\alpha) \, d\alpha \]

\[ R_x(\tau) = \int_{-\infty}^{\infty} R_{xy}(\tau - \alpha) \ast h(\alpha) \, d\alpha \]

\[ S_r(\omega) = \| S_x(\omega) \| H(\omega) \]

\[ Z \]

\[ h \ast h \]
Ex:
Random Telegraph Signal
(Randomized Origin)

\[ R(t) = e^{-2\lambda |t|} \]

\[ S_x(\omega) = a^2 \frac{4A}{4A^2 + \omega^2} \text{rect} \left( \frac{\omega}{2\Omega} \right) \]

\[ R_x(\tau) = \frac{4a^2A}{2\pi} \int_{-\Omega}^{\Omega} \frac{e^{j\omega\tau}}{4A^2 + \omega^2} d\omega \]

\[ \text{var} X = R_x(0) \]
\[ = \frac{4a^2A}{2\pi} \int_{-\Omega}^{\Omega} \frac{d\omega}{4A^2 + \omega^2} \]
\[ = \frac{4a^2A}{4\pi^2} \int_{0}^{\infty} \frac{d\omega}{1 + \left( \frac{\omega}{2A} \right)^2} \]
\[ \omega = \frac{2\lambda}{\omega} \]
\[ = \frac{a^2}{\pi} \int_{0}^{\infty} \frac{d\omega}{1 + \omega^2} \]
\[ = \frac{2a^2}{\pi} \arctan \frac{\Omega}{2A} \]
\[ \frac{\lambda}{2} = 1 \]
Differentiation of $\xi(t)$

Fourier:
\[ \frac{d}{dt} \xi(t) = (j \omega) F(\omega) \]

From relation

\[ S_{\xi'}(\omega) = |j \omega|^2 S_{\xi}(\omega) \]
\[ = \omega^2 S_{\xi}(\omega) \]
\[ = (1) (j \omega)^2 S_{\xi}(\omega) \]

\[ \Rightarrow R_{\xi'}(\tau) = -(\frac{d}{d\tau})^2 R_{\xi}(\tau) \]

We assume that $R_{\xi}$ can be twice differentiated.

O.W., process does not exist

(Ex: Telegraph signal)

Generalizing

\[ R_{\xi^{(n)}}(\tau) = (-1)^n \left( \frac{d}{d\tau} \right)^{2n} R_{\xi}(\tau) \]

if. \( \left( \frac{d}{d\tau} \right)^{2n} \) exists.

\[ \Rightarrow R_{\xi^{(n)}}(\tau) = \left( \frac{d}{d\tau} \right)^{n} R_{\xi}(\tau) \]
Time Invariant

Linear System Response to non-stationary inputs

\[ X(t) \xrightarrow{h(t)} I(t) = L[X(t)] \]

First order statistics:
Fundamental Theorem: \( E[LX] = LEX \)

Thus: Proof:
\[ N_y(t) = E[I(t)] = E \int_{-\infty}^{\infty} X(t-\alpha) h(\alpha) d\alpha \]
\[ = \int_{-\infty}^{\infty} N_x(t-\alpha) h(\alpha) d\alpha \]
\[ = N_x(t) * h(t) \]

Second order statistics
Theorem: \( R_{XX}(t_1, t_2) = L_2 R_{XX}(t_1, t_2) \)

\( R_{XX}(t_1, t_2) = L_1 R_{XX}(t_1, t_2) \)

\[ R_{XX}(t_1, t_2) \xrightarrow{L_2} R_{XX} \xrightarrow{L_1 h(t_1)} R_{XX} \]

\[ R_{XX} = R_X * h(t_2) \]
\[ R_{XX} = R_{XX} * h(t_1) \]
Proof:
\[ y(t_2) = L_{t_2} x(t_2) \]
\[ x(t_1) y(t_2) = x(t_1) L_{t_2} x(t_2) \]
\[ = L_{t_2} x(t_2) x(t_1) \]

End of part 1

\[ y(t_1) = L_{t_1} x(t_1) \]
\[ y(t_2) y(t_1) = L_{t_1} x(t_1) y(t_2) \]
\[ R_{\Xi}(t_1, t_2) = L_{t_1} R_{\Xi}(t_1, t_2) \]

\[ \Xi \text{ in terms of } R_{\Xi} \]
\[ R_{\Xi \Xi} = R_{\Xi} * h \]
\[ R_{\Xi} = R_{\Xi \Xi} * h \]
\[ = R_{\Xi} * h * h \]
\[ = R_{\Xi} * h(t_1) h(t_2) \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\Xi}(t_1, t_2) h(\alpha) h(\beta) \, d\alpha \, d\beta \]

Similarly:
\[ C_{\Xi \Xi}(t_1, t_2) = C_{\Xi}(t_1, t_2) * h(t_2) \]
\[ C_{\Xi}(t_1, t_2) = C_{\Xi \Xi}(t_1, t_2) * h(t_1) \]

Reduced to Station Case for \( R_{\Xi}(t) \)
\[ Z(t) = \sum_{i=1}^{\infty} \delta(t - t_i) \]

\[ n_z = \lambda \quad \Rightarrow \quad R_z(t) = \lambda^2 + \lambda \delta(t) \]

**Proof**

Recall: \( n_x(t) = \lambda t \)

\[ \Rightarrow n_z = \frac{d}{dt} \lambda t = \lambda \]

Recall:

\[ R_{zz}(t,t) = \lambda^2 t, t_2 + \lambda \min(t, t_2) \]

\[ R_{zz}(t_1, t_2) = \frac{\delta R_{zz}(t_1, t_2)}{\delta t_2} \]

\[ = \lambda^2 t_1 + \lambda U(t_1 - t_2) \]

\[ R_{zz}(t_1, t_2) = \frac{\delta R_{zz}(t_1, t_2)}{\delta t_1} \]

\[ = \lambda^2 + \lambda \delta(t_1 - t_2) \]
Ex: Non-stationary white noise

\[ R(t_1, t_2) = 9(t_1) \delta(t_1 - t_2) \]

(for stat. white, \(9(t_1) = \text{const.}\))

\[ R = R \times_t t_2 h(t_2) \]

\[ R = R \times_t t_2 h(t_2) \]

\[ R = R \times_t t_2 h(t_2) \]

\[ R = R \times_t t_2 h(t_2) \]

\[ R = \int_{-\infty}^{\infty} h(t_2 - t_1) \cdot 9(t_1) \cdot h(t_2) \, dt_1 \]
Stochastic Diff. eq.
const coeff:
\[ \sum_{i=0}^{n} a_n X^{(n)}(t) = X(t) \]
Given: \( X^{(i)}(0) = 0 \); \( i = 0 \ldots n-1 \)
Expectation:
\[ \sum_{i=0}^{n} a_n \mathbb{E}[X^{(i)}(t)] = \mathbb{E}[X(t)] \]
\[ \mathbb{E}[X^{(i)}(0)] = 0 \quad (\text{since } X^{(i)}(0) = 0) \]
Autocorrelation:
\[ \sum_{i=0}^{n} a_n \mathbb{E}[X^{(n)}(t_2) X^{(i)}(t)] = X(t) \]
\[ = \sum_{i=0}^{n} a_n \mathbb{E}[X(t_1) X^{(i)}(t_2)] \]
But:
\[ \mathbb{E}[X(t_1) X^{(i)}(t_2)] = \frac{d^i R_{XX}(t_1, t_2)}{dt_2^i} \]
Proof:
\[ \mathbb{E}[X(t_1) (\frac{d}{dt_2})^i X(t_2)] \]
\[ = (\frac{d}{dt_2})^i \mathbb{E}[X(t_1) X(t_2)] \]
Thus:
\[ \sum_{i=0}^{n-1} \sum_{i=0}^{n} a_n \mathbb{E} \left[ (\frac{d}{dt_2})^i R_{XX}(t_1, t_2) \right] = R_{XX}(t_1, t_2) \]
Since:
\[ x(t_1) \mathbb{F}^{(i)}(0) = 0 \]
\[ \frac{d^i R(t_1, t_2)}{dt_2^i} = 0 \quad ; \quad i = 1 \ldots n \]
Similarly:

\[
\sum_{i=1}^{n} a_i \left. \frac{\partial^n R_{yy}(t,t_2)}{\partial t^n} \right|_{t=t_1} = R_{xx}(t_1,t_2)
\]

\[
\left. \frac{\partial R_{yr}(0,t_2)}{\partial t} \right|_{t_1} = 0
\]

(Proof)
\[
\frac{dX}{dt} + \alpha X(t) = \delta(t) \quad X(0) = 0
\]

* is Stot:

\[
EX = \lambda \quad R_{\delta \delta}(t) = \lambda^2 \delta(t)
\]

* Sequence of Poisson Pulses

\[
N_\gamma(t) + \alpha N_\gamma(t) = \lambda \quad N_\gamma(0) = 0
\]

\[
\Rightarrow N_\gamma(t) = \frac{1}{\alpha} (1 - e^{-\alpha t})
\]

Cross Correlation

\[
\frac{dR_{\delta \delta}(t_1, t_2)}{dt_2} + \alpha R_{\delta \delta}(t_1, t_2) = \lambda^2 + \lambda \delta(t_1 - t_2)
\]

Solution:

\[
R_{\delta \delta}(t_1, t_2) = \frac{\lambda^2}{\alpha} \left(1 - e^{-\alpha t_2}\right) + \lambda e^{-\alpha(t_2 - t_1)}
\]

Auto Correlation: \( t_1 < t_2 \)

\[
\frac{dR_{\gamma \gamma}(t_1, t_2)}{dt_1} + \alpha R_{\gamma \gamma}(t_1, t_2) = \frac{\lambda^2}{\alpha} \left(1 - e^{-\alpha t_2}\right) + \lambda e^{-\alpha(t_2 - t_1)}
\]

\[
R_{\gamma \gamma}(0, t_2) = 0
\]

Treat \( t_2 \) as const. Gives for \( t_2 > t_1 \):

\[
R_{\gamma \gamma}(t_1, t_2) = \frac{\lambda^2}{\alpha^2} \left(1 - e^{-\alpha t_2}\right) \left(1 - e^{-\alpha t_1}\right)
\]

\[
+ \frac{\lambda}{\alpha} e^{-\alpha(t_2 - t_1)} \left(1 - e^{-\alpha t_1}\right)
\]

Otherwise \( t_2 < t_1 \), \( R_{\gamma \gamma}(t_1, t_2) = R_{\gamma \gamma}(t_2, t_1) \)
Ergodicity:

\[ S = \frac{1}{2T} \int_{-T}^{T} W(t) \, dt \]

\[ E[s] = \mu = E[W(t)] \]

\[ \overline{S^2} = \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} \text{Cov}(W(t), W(t')) \, dt \, dt' \]

\[ \overline{S^2} = \frac{1}{2T} \int_{-T}^{T} (2T - 1T) R(\tau) \, d\tau \]

\[ \text{var} S = \frac{1}{2T} \int_{-T}^{T} (2T - 1T) R(\tau) \, d\tau - \mu^2 \]

\[ = \frac{1}{2T} \int_{-T}^{T} (2T - 1T) \left[ R(\tau) - \mu^2 \right] \, d\tau \]

\[ = \frac{1}{4T} \int_{0}^{2T} (2T - 1T) (R(\tau) - \mu^2) \, d\tau \]

Ex: For telegraph (randomized origin)

\[ R(\tau) = e^{-2\lambda |\tau|} \]

\[ \text{var} S = \frac{1}{2\lambda T} - \frac{1 - e^{-2\lambda T}}{8\lambda^2 T^2} \]
**TIME AVERAGES**

\[
\langle X \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) \, dt
\]

**STATISTIC**

\[
\langle R(\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t+\tau) X(t) \, dt
\]

\[
\langle \sigma_X^2 \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [X(t) - \langle X \rangle]^2 \, dt
\]

Ergodicity in the mean. Requires two criteria:

1. \( E[\langle X \rangle] = \mu \)
2. \( \text{var} \langle X \rangle = 0 \)

Then, \( \mu \) can be estimated determined with probabilistic one stochastic process.

Equivalently:

1. \( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) \, dt = E[X] = \mu \) - true if WSS
2. \( \lim_{T \to \infty} \frac{1}{T} \int_{0}^{2T} (1 - \frac{\tau}{2T}) [R(\tau) - \mu^2] \, d\tau = 0 \)

For telegraph:

1. WSS
2. \( \frac{1}{\lambda T} - \frac{1 - e^{-2\lambda T}}{8\lambda^2 T^2} \to 0 \) as \( T \to \infty \)

\[ \therefore \text{Ergodic in the mean} \]
Sufficient Condition for Ergodicity in the mean:

1. If \( X(t) \) is WSS and \( \int_{\mathbb{R}} |C(\tau)| \, d\tau < \infty \), then \( X \) is mean ergodic.

   **Proof:**
   
   \[
   \frac{1}{2T} \int_{-T}^{T} C(\tau) \left( 1 - \frac{1}{2T} \right) \, d\tau < \frac{1}{2T} \int_{-T}^{T} |C(\tau)| \, d\tau
   \]
   
   If finite, then \( T \to \infty \).

2. If \( C(0) < \infty \) and \( C(\tau) \to 0 \) as \( |\tau| \to \infty \), then \( X(t) \) is mean-ergodic.

   **Note:** \( X(t+\tau) \) and \( X(t) \) are uncorrelated for \( \tau \to \infty \)

   **Proof in text.**

---

**Example:**

\[
\mathbb{E}[\mathbb{E}(X(t))] = A
\]

\[
\frac{1}{2T} \int_{-T}^{T} X(t) \, dt = A \neq \mathbb{E}[A]
\]

for all \( T \),

not mean ergodic.
Stationary Ex White Noise

\[ C(t_1, t_2) = q \delta(t_1 - t_2) \]

or

\[ C(\tau) = q \delta(\tau) \]

If zero mean

\[ S(\omega) = q \iff \text{why called white} \]

From condition 1:

\[ \int_{-\infty}^{\infty} |C(\tau)| d\tau = q < \infty \]

\[ \therefore \text{process is mean ergodic} \]

Problem: \[ \sigma^2 = q \delta(0) = \infty \]

Uncorrelated.
Correlation-Ergodic Process

\[ \langle R(\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t+\tau) \overline{X}(t) \, dt \]

Define

\[ E\langle R(\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X(t+\tau) \overline{X}(t)] \, dt \]

\[ = R(\tau) \]

Define

\[ Z_{\gamma}(t) = X(t+\tau) \overline{X}(t) \]

If \( Z_{\gamma}(t) \) is mean-ergodic, then \( X(t) \) is Correlation Ergodic.

(Requires fourth-order statistics).
Generating Random Numbers

1. Use Table
2. Pseudo-Random numbers

Congruence Method of generating pseudo-random numbers

\[ X_{n+1} = (a \cdot X_n + b) \mod T \]

\( b \not\equiv T \) should be relatively prime

Example: \( a = 3 \), \( b = 11 \), \( T = 1 \)

(R) Seed \( X_0 = 1 \) \hspace{1cm} \text{NOTE: CAN GET FROM RANDOM # TABLE}

\[
\begin{align*}
X_1 &= 0.1415926 \\
X_2 &= 0.641592654 \\
X_3 &= 0.103981635 \\
X_4 &= 0.2975656
\end{align*}
\]

Can Show:

\[
\rho_s = E[X_n X_{n+s}] = 1 - 6 \frac{b_s}{T} (1 - \frac{b_s}{T}) + \epsilon
\]

\[ a_s = a_s^5 \pmod{T} \]

\[ b_s = (\sum_{n=b}^{s-1} a^n) b \pmod{T} \]

\[ |\epsilon| < a_s/T \]

\[ H_p \quad X_{n+1} = \text{Fr}(X_n + 11)^5 \]
1. How can we generate

\[ Y = X + \frac{1}{2} \]

2. What about a dice roll?

\[ D_n = \text{Int} \left[ 6 \cdot X_n + 1 \right] \]

3. How about the sum of two dice?

\[ D_n + D_{n+1} = \text{sum} \]

3. Gaussian R.V.
(a) can find the \( Y_n = g(X_n) \) is Gaussian (ugly)
(b) Central limit theorem

\[ Y = X_1 + X_2 + X_3 + \ldots + X_N \]

\[ \approx \text{Mean} = \frac{N}{2}, \quad \text{var} = \frac{N}{12} \]

(c) Compute \( X_n \frac{1}{2} X_{n+1} \)

then

\[ Y_n = \left( -2 \ln X_1 \right)^{\frac{1}{2}} \cos 2\pi X_1 \]

\[ Y_{n+1} = \left( -2 \ln X_1 \right)^{\frac{1}{2}} \sin 2\pi X_2 \]

\( Y_n \frac{1}{2} Y_{n+1} \) are zero mean

- unit variance normal r.v.'s.
<table>
<thead>
<tr>
<th>Table 26.11</th>
<th>2500 FIVE DIGIT RANDOM NUMBERS</th>
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Compiled from Rand Corporation, A million random digits with 100,000 normal deviates. The Free Press, Glencoe, Ill., 1955 (with permission).
ANALOG TECHNIQUES

Low Pass Response

\[ X(t) \rightarrow \text{LPF} \rightarrow Y(t) \]

(Real \ $\xi$ \ Stationary)

Assume \ \( R(\tau) = 0 \) for \ \( |\tau| > a \)

\[ R(\tau) \]

\[ a = \text{correlation length} \]

Also, assume \ \( h(t) \approx \text{constant} \) \ in \ any \ correlation \ length \ interval:

\[ h(t) \]

Correlation length intervals

Then

\[ \int_{-\infty}^{\infty} h(t) R(t-\alpha) \, dt \approx h(\alpha) \int_{-\infty}^{\infty} R(t-\alpha) \, dt \]
Then:

$$E[y^2(t)] = q \cdot E$$

where:

$$q = \int_{-a}^{a} R(t) \, dt$$

$$\frac{1}{2} E = \int_{-\infty}^{\infty} |h(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \, d\omega$$

Proof: Recall:

$$R_{yy}(t_1, t_2) = \iint R_x(t_1 - \alpha, t_2 - \beta) \, h(\alpha) \, h(\beta) \, d\alpha \, d\beta$$

Thus:

$$E[y^2] = R_{yy}(t, t) = \iint R_x(t - \alpha, t - \beta) \, h(\alpha) \, h(\beta) \, d\alpha \, d\beta$$

$$= \iint R_x(t - \alpha - t + \beta) \, h(\alpha) \, h(\beta) \, d\alpha \, d\beta$$

$$= \iint R_x(t - \alpha) \, h(\alpha) \, h(\beta) \, d\alpha \, d\beta$$

on this strip, $$h(\alpha) = h(\beta)$$ by assumption (since $$\alpha, \beta$$ are "close")
Thus

\[ E[X^2(t)] = \mathbb{E} \left[ \int_{-\infty}^{\infty} h^2(\alpha) R(B-\alpha) d\beta d\alpha \right] \\
= \int_{-\infty}^{\infty} h^2(\alpha) \mathbb{E} R(B-\alpha) d\alpha \\
= \text{Eq} \]
Analog Techniques

Estimate mean of process, \( X(t) \).

\[
\begin{align*}
W(\omega) &= \int_{-\infty}^{\infty} w(t) e^{-j\omega t} dt \\
Y(t) &= \int_{-\infty}^{\infty} X(t - \alpha) w(\alpha) d\alpha
\end{align*}
\]

Output:

Q: What is \( W(\omega) \) such that \( Y(t) = \eta_x \)?

A: ① \( \eta_y = \eta_x \)

2. \( \sigma_y << \sigma_x \)

① Set \( W(0) = 1 \) since \( \eta_y = W(0) \eta_x \)

② Must evaluate \( \sigma_y \)
Assume bandwidth, \(\omega_c\), of LPF is "small" w.r.t. so that \(w(t)\) is "constant" w.r.t. correlation length of \(\xi(t)\)

\[
\begin{align*}
R(\tau) & \quad \tau \\
\text{Correlation length} & \\
\end{align*}
\]

Then, recall:

\[
E[\xi^2] = q \mathcal{E} \\
\text{Thus } E[\xi^2] = q \mathcal{E} - n_y^2 = \hat{q} \mathcal{E} \quad (\hat{q} = \int_{-\infty}^{\infty} C_\xi(\tau) d\tau) \\
E = \int_{-\infty}^{\infty} w^2(t) dt
\]

Then, second condition becomes:

\[
\hat{q} \mathcal{E} \ll n_X^2
\]

Thus, in summary:

\(\Theta\)
By Parseval's Theorem:

\[ E = \int_{-\omega_c}^{\omega_c} \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} |W(\omega)|^2 d\omega \]

Assume \( \omega_0 = \text{Bandwidth} \)

Thus, is summary, wish:

1. \( W(0) = 1 \)
2. \( \omega_c < \frac{\pi \omega_c^2}{4} \)
Correlometer

\[ H(\omega) = e^{j\omega} \]

\[ x(t) \]

\[ x(t) x(t-\lambda) \]

\[ LP \]

\[ y(t) \approx R(\lambda) \]
Finding Density Function from Single Sample

Use ZNL

Then:

\[ Z(t) = \begin{cases} \int \frac{1}{\Delta x} & ; x_0 < X(t) < x_0 + \Delta x \\ 0 & ; \text{otherwise} \end{cases} \]

\[ E[Z(t)] = f(x_0) \]

= average time \( X(t) \) is 'twixt \( x_0 \) \( \frac{1}{2} \) \( x_0 + \Delta x \)

Must use \# stages for entire process.
Solution

1. \( \Phi_X(\omega) = 1 + \left( \frac{\omega}{a} \right)^2 \Rightarrow F_X(\omega) = \frac{1}{\sqrt{1 + (\frac{\omega}{a})^2}} \)

\( \Phi'(\omega) = -\frac{2\omega/\alpha^2}{1 + (\omega/\alpha)^2} \Rightarrow \Phi(0) = 0 \)

\( \Phi''(\omega) = \frac{1 + (\alpha/\omega)^2}{[1 + (\omega/\alpha)^2]^2} \Rightarrow \Phi''(0) = -\frac{2}{a^2} = -\text{var} \Rightarrow \text{var} = \frac{1}{\alpha^2} \)

2. \( Y = \Phi(X) \), Find \( f_{X|Y}(x,y) \) for \( X > 0 \) \& \( Y > 1 \)

Line mass: \( x \)

\[ F_X(x,y) = \begin{cases} 0 & x > 0, y > 1 \\ 1 & \text{otherwise} \end{cases} \]

3. \( X \sim e^{-X}U(x) \)

Rayleigh

Transformation for increasing \( g(x) \)

\[ f_Y(y) = \left| \frac{dg^{-1}(y)}{dy} \right| f_X(g^{-1}(y)) \]

\[ y e^{-\frac{x^2}{2}} = \left[ \frac{d}{dy} g^{-1}(y) \right] e^{-g^{-1}(y)} \Rightarrow x = \frac{y}{2} = g^{-1}(y) \text{ works } \Rightarrow y = g^{-1}(y) = \sqrt{2x} \]

4. \( f_{X|Y}(x,y) = \int \gamma e^{-\gamma(x+y)} U(x)U(y) \)

(a) \( y = ? \)

\[ y = 0 \Rightarrow 0 \int_0^\infty \int_0^\infty y e^{-\gamma(x+y)} dxdy = 0 \int_0^\infty \gamma e^{-\gamma y} dy = \gamma \int_0^\infty e^{-\gamma y} dy = \gamma \Rightarrow \gamma = 1 \]

(b) \( f_Y(y) = \int_0^\infty f_{X|Y}(x,y) dx = \int_0^\infty y e^{-\gamma x} e^{-\gamma y} dy = \gamma^2 \int_0^\infty e^{-\gamma y} dy = \gamma^2 \]

5. \( \Theta_X(\omega) = \int \Phi_X(\omega) \)

\[ d \Theta_X(\omega) /d\omega = \Phi_X(\omega) /\Phi_X(\omega) \Rightarrow \Theta'(0) = \Phi'(0) /\Phi(0) = \text{JACUNDEP \ since } \Phi(0) = 0 \]

6. \( Z = X/Y \) \& \( F_Z(z) = P_r[X/Y \leq z] = P_r[X/Y \leq 1] \)

\[ P_r[X/Y \leq z] = \int -x/z f_{X|Y}(x,y) dxdy \]

\( \Theta_0 \) integral

7. Tchebycheff Inequality (for zero mean):

\[ P_r[|X| \geq \varepsilon] \leq \sigma^2 /\varepsilon^2 \]

\[ \sigma = \varepsilon \]

We want \( P_r[|X| \leq T] \geq 1 - e^{-T^2/2} = 1 - \frac{1}{10} \text{ since } \sigma = 1 \)

\[ \frac{1}{T^2} = 0.01 \Rightarrow T^2 = 100 \Rightarrow T = 10 \]
EE505 Final Examination

Robert J. Marks II

August 20, 1997; 2:20 to 4:20 PM

The examination is closed book and closed notes. No calculators are allowed. Each student is allowed three sheets of notes. All problems are weighted equally. Work must be done in ink.

All work will be done in a test booklet. No scratch paper is needed.

"And I trust that you will discover that we have not failed the test."
Corinthians 13:6 (English-NIV)

1. Let $X$ and $Y$ be independent random variables and let $Z = X + Y$. Prove or disprove the following propositions.

(a) $Z = X + Y$

(b) $Z^2 = X^2 + Y^2$

(c) $\text{var}(Z) = \text{var}(X) + \text{var}(Y)$.

(d) $\text{var}(aZ) = a^2 \text{var}Z$.

Thus $\text{var} Z = \text{var} X + \text{var} Z \Leftrightarrow \text{True}$
2. 

\[ Y = \frac{1}{N} \sum_{k=1}^{N} X_k^2 \]

where the \( X_k \)'s are i.i.d. random variables with probability density function 

\[ f_X(x) = e^{-x} \mathbb{1}(x) \]

Estimate the probability density function for the random variable \( Y \) when \( N \) is large.\(^1\)

Let 

\[ Z_k = X_k^2 \Rightarrow \bar{Z}_k = \frac{1}{N} \sum_{k=1}^{N} Z_k = \frac{1}{N} \sum_{k=1}^{N} X_k^2 = \frac{1}{N} \sum_{k=1}^{N} 1 = 1 \]

\[ \bar{Z}_k = \frac{1}{N} \sum_{k=1}^{N} Z_k = 3! = 6 \Rightarrow \text{var} \ Z_k = 5 \]

From problem \# 1 

\[ \sum_{k=1}^{N} \bar{X}_k = \frac{1}{N} \sum_{k=1}^{N} X_k^2 = \frac{1}{N} \sum_{k=1}^{N} 1 = N \]

\[ \text{var} \ Z_k = \text{var} \ \frac{1}{N} \sum_{k=1}^{N} Z_k = \frac{1}{N} \text{var} \ Z_k = \frac{1}{N} \sum_{k=1}^{N} 5 = 5N \]

By Central Limit Theorem:

\[ \frac{1}{N} \sum_{k=1}^{N} \bar{X}_k \sim \mathcal{N} (N, \sqrt{5N}) \]

and

\[ Y = \frac{1}{N} \sum_{k=1}^{N} Z_k \sim \mathcal{N} \left( 1, \frac{5}{N} \right) \]

or:

\[ f_Y(y) \approx \frac{1}{\sqrt{2\pi N} \sqrt{5/N}} \exp \left( - \frac{(y-1)^2}{5/N} \right) \]

\(^1\)Recall from the last test that the \( n \)th moment of each \( X_k = \Theta(n+1) \)
3. A total of $N$ i.i.d. Bernoulli trials with probability of success $p$ are performed. The outcome of trial $m$, the random variable $X_m$, is set to one if there is a success and zero otherwise. We form the sum

$$Y = \sum_{m=1}^{N} X_m.$$ 

Evaluate the exact probability density function for the random variable $Y$.

This is simple binomial R.V.

$$P_k = P_r[I = k] = \binom{N}{k} p^k q^{N-k}; q = 1 - p$$

$$f_I(y) = \sum_k P_k \delta(y - k)$$

$$= \sum_{k=0}^{N} \binom{N}{k} p^k q^{N-k} \delta(y - k)$$
4. The Weibull random variable $Y$ with positive parameters $A$ and $B$ is

$$F_Y(y) = \left[1 - \exp\left(-\frac{y^B}{A}\right)\right] U(y).$$

Let $X$ be a uniform random variable on the interval $(0, 1)$. Given $A$ and $B$, find a random variable transformation, $Y = g(X)$, to produce a Weibull random variable.

$$Y = \mathcal{G}(X)$$

$$y = F_X^{-1}(x) = \mathcal{G}(x)$$

OR

$$x = F_Y(y) = \left[1 - e^{-\left(\frac{y}{A}\right)^B}\right]$$

$$e^{-\left(\frac{y}{A}\right)^B} = 1 - x$$

$$\left(\frac{y}{A}\right)^B = -\ln(1 - x)$$

$$\frac{y}{A} = \left[-\ln(1 - x)\right]^{\frac{1}{B}}$$

$$y = A \left[-\ln(1 - x)\right]^{\frac{1}{B}} = \mathcal{G}(x)$$
5. A random variable has a probability density function of

\[ f_X(x) = e^{-x}u(x) \]

We take i.i.d. samples from this distribution until we get a number between zero and one - and then stop. Call this last random variable \( Y \). Evaluate the probability density function of \( Y \).

\[
F_Y(y) = \Pr[ \frac{X}{A} \leq y ] = \Pr[ \frac{X}{A} \leq x | 0 \leq x \leq 1 ]
\]

Thus

\[
f_Y(y) = \begin{cases} \frac{A}{1-e^{-1}} e^{-xy} & ; 0 < y < 1 \\ 0 & ; \text{o.w.} \end{cases}
\]

\[
1 = A \int_0^1 e^{-x} \, dx = A \left[ e^{-x} \right]_0^1 = A \left( 1 - e^{-1} \right)
\]

\[
\Rightarrow A = \frac{1}{1-e^{-1}}
\]

\[
\therefore f_Y(y) = \begin{cases} \frac{e^{-y}}{1-e^{-1}} & ; 0 \leq y \leq 1 \\ 0 & ; \text{o.w.} \end{cases}
\]
6. Let $X$ be a zero mean normal random variable with variance $\sigma^2$. Let $Y = X$ when $X$ is positive and let $Y = 1$ otherwise. Evaluate and sketch the probability density function for $Y$. 

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \cdot \mathbb{1}(y-1) + \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \mathbb{1}(y)$$
7. A joint probability density function is defined by

\[ f_{XY}(x, y) = \begin{cases} 
A & ; |y| \leq e^{-x} \text{ and } y \geq 0 \\
0 & ; \text{otherwise}
\end{cases} \]

(a) Evaluate \( A \).

(b) Evaluate the marginal distribution, \( f_Y(y) \).

\[ f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \]

\[ = A \int_{0}^{\infty} e^{-x} \, dx \]

\[ = A \left[ -e^{-x} \right]_{0}^{\infty} \]

\[ = \frac{1}{2} \ln |y| ; \quad |y| < 1 \quad \text{and} \quad 0 \quad \text{o.w.} \]
1. d = dog fish twice as hungry  
   ⇒ Effective # of dog fish = 60,000
   \[ p_d = \frac{6}{6+2+1} = \frac{6}{9} = \frac{2}{3} \]
   c = catfish \(\Rightarrow p_c = \frac{1}{9}, \ p = perch \Rightarrow p_p = \frac{1}{9} \)
   \[ \text{Prob} [k_d = 3, k_c = 0, k_p = 1] = \frac{4!}{3!0!1!} \left( \frac{2}{3} \right)^3 \left( \frac{1}{9} \right)^0 \left( \frac{1}{9} \right) \approx 0.263 \]

2. \( m_n = E[X^n] = \frac{1}{2c} \int_{-c}^{c} x^n dx = 0 \text{ if } n \text{ is odd} \)
   \[ = \frac{1}{2c} \cdot 2 \int_0^c x^n dx \text{ if } n \text{ is even} \]
   \[ = \frac{1}{2} \frac{c^{n+1}}{n+1} \text{ if } n \text{ is even} \]
   \[ : m_n = \begin{cases} 0 & \text{n odd} \\ \frac{c^n}{n+1} & \text{n even} \end{cases} \]

3. \( X \sim N(3, 1) \)
   \[ P_r[2 \leq X \leq 4] = \frac{P_r[2 \leq X \leq 4]}{P_r[X > 2]} \]
   \[ = \frac{P_r[2 \leq X \leq 4]}{P_r[X > 2]} \]
   \[ = \frac{P_r[-1 \leq \frac{X-3}{1} \leq 1]}{P_r[-1 \leq \frac{X-3}{1} \leq \infty]} \]
   \[ = \frac{\text{erf}(1) - \text{erf}(-1)}{\text{erf}(\infty) - \text{erf}(-1)} = \frac{2 \text{erf}(1)}{\frac{1}{2} + \text{erf}1} \]
   \[ = \frac{1}{2 \text{erf}(1) + 1}, \ \text{erf}(1) = 0.34134 \]
   \[ = 0.81 \]

4. (a) \( \Phi(o) = 1 = \exp(e^b - a) \Rightarrow a = e^b \) or \( b = \ln a \)
   (b) \( \psi(s) = \ln \Phi = e^{s+b} - a \)
   \[ \frac{d\psi}{ds} = e^{b+s} \Rightarrow \psi'(o) = \lambda = e^{b} = a \]
   \[ \frac{d^2\psi}{ds^2} = e^{b+2s} \Rightarrow \theta^2 = 2 = \psi''(o) \]
5. \[ E[X^n] = \frac{1}{2} \int_{-1}^{1} e^{2nx^2} \, dx \]
\[ 2nx^2 = \frac{2}{n} z^2 \]
\[ \Rightarrow z = 2\sqrt{n} x \Rightarrow dx = \frac{1}{2\sqrt{n}} \, dz \]
\[ E[X^n] = \frac{1}{2} \int_{-2\sqrt{n}}^{2\sqrt{n}} e^{-\frac{z^2}{2}} \frac{1}{2\sqrt{n}} \, dz \]
\[ = \frac{1}{\sqrt{2\pi n}} \cdot 2 \int_{0}^{\sqrt{n}} e^{-\frac{z^2}{2}} \, dz \]
\[ = \frac{1}{\sqrt{2\pi n}} \text{erf}(2\sqrt{n}) \]
\[ \Rightarrow E(Y) = \frac{1}{\sqrt{2\pi}} \text{erf}(2) = \frac{1}{\sqrt{2\pi}} \cdot 0.47726 = 0.60 \]
\[ E(Y^2) = \frac{1}{\sqrt{2\pi}} \text{erf}(2\sqrt{2}) = \frac{\sqrt{\pi}}{2} \cdot 0.498 = 0.44 \]
\[ \Rightarrow \text{var} Y = 0.084 \]

6. \( y = g(x) = x^n \Leftarrow \text{strictly } \uparrow \text{ for odd } n \)
\[ \Rightarrow x = y^{\frac{1}{n}} = g^{-1}(y) \]
\[ \frac{dg^{-1}(y)}{dy} = \frac{1}{n} y^{\frac{1}{n} - 1} \]
\[ \Rightarrow f_X(y) = \begin{cases} \frac{1}{2nc} y^{\frac{1}{n} - 1} & ; 1y1 \leq c^n \\ 0 & ; \text{otherwise} \end{cases} \]
FINAL EXAMINATION

Name........................................
Score........................................

INSTRUCTIONS

1. Mail your completed examination to:

   Dr. Robert J. Marks II
   16515 Ashworth Ave. N.
   Seattle, WA 98133

   Submit your work stapled and in order with this page as the cover sheet. Your envelope must be postmarked no later than June 8, 1987.

2. Since this is a take home exam, neatness and clarity of your presentation are taken into account.

3. Your lowest quiz grade will be dropped. In addition, you may elect to replace a second quiz score with your score on the corresponding problem on this test (i.e. problem 1 for quiz 1, etc.). Any ambiguity or unclarity in this request will result in it being ignored. Make your request, if any, here:

   ........................................................................................................

4. After completing the exam, please sign the following:

   "I have received no outside (human) help on this examination or, if I have, the names of the people I have consulted are listed below my signature".

   X...................................................
   date............................................
PROBLEMS:

1. Four integer numbers are to be encoded in a generalization of a Hamming code. We have the standard table:

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The 4 integer numbers are labeled 3, 5, 6, 7. Integer 1, then, is the sum of integers 3, 5 and 7. Integer 2 is the sum of 3, 6 and 7 and integer 4 is the sum of 5, 6 and 7. Thus, the integers 8, 10, 9, 1 would be coded as 19, 18, 8, 20, 10, 9, 1. Suppose, then, 4 other integers were so encoded and sent over a noisy channel. At the receiver, we decode 7, 8, 1, 9, 2, 5, 4. One of these integers is wrong. Find out which one it is and correct it.

2. A bandlimited signal, \( f(t) \), has a maximum frequency of \( B \) hertz. Assume that we sample in excess of \( 2B \) samples per second. The sample taken at the origin, \( f(0) \), however, is lost. Show how we can regain this lost sample from those remaining. (Hint: What happens to the replicated spectrum when the sample at the origin is lost?)

3. The signal \( 8 \cos(\omega t) \) is sent over a linear time-invariant distortionless channel. The received signal is \( 4 \sin(\omega t) \). What is the received signal when the transmitted signal is \( 4 \sin(\omega t) \)?

4. A binary string of numbers codes a logic 1 as an isosceles triangle of height \( A \) and duration \( T \). A logic zero is the negative of this. Assuming an equal density of ones and zeros, what is the power spectral density of this encoding technique?

5. White gaussian noise has a uniform power spectral density of height \( N/2 \). What percentage of the time does the noise waveform exceed one?

6. A transmitted DSB signal undergoes a square-law nonlinear transformation. That is, the received modulated signal is the square of what it should be. Is the signal degraded beyond recovery? If not, please explain a process by which it can be regained.

7. Bill the radioman says he can use an envelope detector to demodulate FM. He says you can run the modulated signal through a differentiator and then the envelope detector. Assuming that Bill has an RF differentiator, is he right?
Instructions:

1. Do all of your work in this test booklet.
2. You are allowed one sheet of written notes and a calculator. A table of erf function values is given below.
3. Each problem is worth 20 points.

Table 3-1 \[ \text{erf} \ x = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2} \, dy = \frac{G(x) - \frac{1}{2}}{2} \]

<table>
<thead>
<tr>
<th>x</th>
<th>erf x</th>
<th>x</th>
<th>erf x</th>
<th>x</th>
<th>erf x</th>
<th>x</th>
<th>erf x</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.01994</td>
<td>0.80</td>
<td>0.28814</td>
<td>1.55</td>
<td>0.43943</td>
<td>2.30</td>
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<td>0.03983</td>
<td>0.85</td>
<td>0.30234</td>
<td>1.60</td>
<td>0.44520</td>
<td>2.35</td>
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<tr>
<td>0.15</td>
<td>0.05962</td>
<td>0.90</td>
<td>0.31594</td>
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<td>0.45053</td>
<td>2.40</td>
<td>0.49180</td>
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<td>0.20</td>
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<td>0.32894</td>
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<td>0.45543</td>
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<td>0.45994</td>
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<td>2.00</td>
<td>0.47726</td>
<td>2.75</td>
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<td>0.40320</td>
<td>2.05</td>
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<td>0.49744</td>
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<td>2.25</td>
<td>0.48778</td>
<td>3.00</td>
<td>0.49865</td>
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</table>
Problem 1: In Lake Washington there are 10,000 Catfish, 20,000 Perch and 30,000 Dogfish. The Dogfish are twice as hungry as the Catfish and the Perch. Olie went fishing and caught 4 fish. All the fish were either Catfish, Perch, or Dogfish. After Olie caught each fish, he set it free. What is the probability that three of the fish were Dogfish and one was a Perch? *Express your answer as a single number.*
Problem 2: Compute all of the moments of a random variable that is uniform over the interval of $-c$ to $c$. 
Problem 3: The diameters of apples grown in eastern Washington is modeled as a Gaussian or normal random variable with a mean of three inches and a standard deviation of one inch. A sorting machine rejects those apples whose diameter is less than two inches. After sorting, what is the probability that an apple has a diameter between two and four inches? Express your final answer as a number.
Problem 4: A random variable has a moment generating function of \( \exp[\exp(s+b)-a] \) where \( a \) is a given parameter.

(a) What is \( b \)?
(b) Compute the mean and the variance of this random variable.
Problem 5: The random variable $X$ is uniform on the interval of minus one to one. Let $Y = \exp(-2X^2)$. Compute a numerical value for the mean and variance of $Y$. 
Problem 6: Let $N$ be a positive odd integer other than one and let $X$ denote a random variable that is uniform over the interval of $-c$ to $c$. Compute the probability density function for $Y = X^N$. 
mini quiz one:

Baseball player A has a batting average of 0.300. Batter B's is 0.200. Manager C rolls a die. If the result is a three or a six, then batter B bats. Otherwise, batter A bats. The batter gets a hit. What is the probability that it was batter A?
mini quiz one:

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\[ P(H) = P(H|A)P(A) + P(H|B)P(B) \]
\[ = (0.3)\left(\frac{2}{3}\right) + (0.2)\left(\frac{1}{3}\right) \]
\[ = 0.26667 \]

\[ P(H,A) = P(H|A)P(A) + P(A|H)P(H) \]
\[ \Rightarrow P(A|H) = \frac{P(H|A)P(A)}{P(H)} \leq \text{Bayes} \]
\[ = \frac{(0.300)^{2/3}}{0.26667} = 0.75 \]
\[ \text{or } 75\% \]
mini quiz #2

You receive 3 cards from a standard deck of 52. Find the probability that:
(a) ...at least two are clubs.
(b) ...at least two are of the same suit.
(c) ...one is an ace and two are kings.
mini quiz #2

with replacement

You receive 3 cards from a standard deck of 52. Find the probability that:
(a) ...at least two are clubs.
(b) ...at least two are of the same suit.
(c) ...one is an ace and two are kings.

(a) Three repeated Bernoulli Trials.

\[ p = \frac{1}{4} \]

\[ P_r[2 \text{ are clubs or three are clubs}] 
= \left( \frac{3}{4} \right)^2 q + \left( \frac{3}{4} \right)^3 q^0 = 3\left( \frac{1}{4} \right)^2 \left( \frac{3}{4} \right) + \left( \frac{1}{4} \right)^3 = 0.156 \]

(b) \[ P_r[\text{at least two are of the same suit}] 
= P_r[\text{at least 2 are clubs}] + P_r[\text{"" diamonds}] + P_r[\text{"" hearts}] + P_r[\text{"" spades}] 
= 4 \times 0.156 = 0.625 \]

(b) The events are mutually exclusive.

(c) Partition: \( A_1 = \text{ace}, A_2 = \text{king}; A_3 = \text{other} \)

\[ p_1 = \frac{1}{13}, p_2 = \frac{1}{13}, p_3 = \frac{11}{13} \]

\[ P_r[k_1 = 1, k_2 = 2, k_3 = 0] = \frac{3!}{1!2!0!} p_1^1 p_2^2 p_3^0 
= 3 \left( \frac{1}{13} \right)^3 = 0.0013655 \]
FINAL EXAMINATION

INSTRUCTIONS:
1. Mail your exam with this page as a cover sheet to:
   Dr. Robert J. Marks II
   16515 Ashworth Ave. N.
   Seattle, WA 98133
2. If you want the graded exam mailed directly to you, include a self addressed stamped envelope. Otherwise, the exam will be returned to Cogswell.
3. Please sign the following:
   "I have neither received nor given any information concerning this examination or if I have received or given information, the details of this exchange are given on the back of this page."

   X
   (sign)

   (print your name) (date)

4. neatness and clarity of the presentation of your results will be taken into account.

EXAMINATION PROBLEMS:

1. A family has three children, none of which are twins or triplets. What is the probability that all three are born on the same day of the year? What is the probability that all three are born on the same day of the year and all three are boys? What is the probability that two of the 3 are boys both born on the same day of the year?

2. Problem 3-5 in Papoulis (p. 90).

3. A Poisson process with parameter $\lambda = 2$ occurances per hour is observed for one half of an hour. What is the probability that the number of occurances exceeds two given that the number of occurances exceeds one? Give a single number for your final answer.

4. $X$ and $Y$ are independent random variables. Both are distributed uniformly on the interval from zero to one. Let $Z = XY$.
   (a) Compute $f_Z(z)$.
   (b) Find $\Pr(Z \leq 1/2)$.

5. Problem 7-2 in Papoulis (p. 170).
7. Problem 9-1 in Papoulis (p. 258)
(a) \( P = \frac{1}{(365.25)^2} \approx 7.5 \times 10^{-6} \)

\( P = \frac{1}{(365.25)^2} \approx 0.00205 \) or 1 chance in 488.34

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(b) The date of birth is independent of sex. Here

\( P = \frac{1}{(365.25)^2} \approx 0.00205 \) or 1 chance in 488.34

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(c) Children = A, B, C

\( P = \frac{1}{(365.25)^2} \approx 0.00205 \) or 1 chance in 488.34

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\( P = \frac{1}{(365.25)^2} \approx 0.00205 \) or 1 chance in 488.34

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Thus: 

\[ P = \begin{cases} 0.99986.5 & n = 600 \\ 1 - \frac{e^{-1} \left( \frac{0.1}{0.1 + 0.1} \right)^n}{n!} & n \geq 1 \end{cases} \]
\[ F_z(z) = \Pr \left[ X Y \leq z \right] = \Pr \left[ \frac{z}{z} x + \frac{z}{z} y \leq z \right] = \int_{x=0}^{z} \int_{y=0}^{z-x} \frac{1}{z} \, dy \, dx = \int_{x=0}^{z} \left[ \frac{z-x}{z} \right] \, dx = \int_{x=0}^{z} \left( 1 - \frac{x}{z} \right) \, dx = \left[ x - \frac{x^2}{2z} \right]_{x=0}^{z} = z - \frac{z^2}{2z} = z - \frac{z}{2} = \left\{ \begin{array}{ll}
\frac{z}{2} & \text{if } 0 \leq z \leq 1 \\
0 & \text{otherwise}
\end{array} \right. \]

\[ f_z(z) = \frac{d}{dz} F_z(z) = \frac{1}{2} \quad \text{for } 0 \leq z \leq 1 \]

\[ P_r \left[ Z < \frac{1}{2} \right] = \int_{0}^{\frac{1}{2}} f_z(z) \, dz = \int_{0}^{\frac{1}{2}} \frac{1}{2} \, dz = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]

\[ F_z(\frac{1}{2}) = \int_{0}^{\frac{1}{2}} f_z(z) \, dz = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]

\[ F_z(1) = \int_{0}^{1} f_z(z) \, dz = \int_{0}^{1} \frac{1}{2} \, dz = \frac{1}{2} \times 1 = \frac{1}{2} \]

\[ f_z(1) = \frac{d}{dz} F_z(1) = \frac{1}{2} \quad \text{for } 0 \leq z \leq 1 \]

\[ \int_{0}^{\frac{1}{2}} f_z(z) \, dz = \frac{1}{4} \]

\[ f_z(\frac{1}{2}) = \frac{1}{2} \]

\[ F_z(1) = \frac{1}{2} \]

\[ f_z(1) = \frac{1}{2} \]
5. \[ E[z] = E[(x-y)U(x-y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-y)U(x-y)e^{-x}U(x)e^{-y}dydx \]
   \[ = \int_{y=0}^{\infty} \left[ \int_{x=y}^{\infty} (x-y)e^{-x}dx \right] e^{-y}dy = \frac{1}{2} \]

6. \[ n_r = 500 \]
   \[ \sigma_r^2 = \frac{1}{100} \int_{r=0}^{50} r^2 dr = \frac{50^2}{3} \]
   \[ X = r_1 + r_2 + r_3 + r_4 \]
   \[ n_x = 4 \cdot 500 = 2000 \]
   \[ \sigma_x^2 = 4 \cdot \frac{50^2}{3} = \frac{10^4}{3} \]
   From CLT,
   \[ X \sim N(2000; \frac{10^2}{13}) \]
   Thus
   \[ P_r[1900 < X < 2100] = 2 \cdot \text{erf} \left( \frac{100}{10^2/\sqrt{13}} \right) 
   = 0.9169 \]

7. \[ P_r(X(t) = 2t) = \frac{1}{2} = P_r[X(t) = \sin \pi t] \]

(a) \[ \mathbb{E}[X(t)] = \frac{1}{2}(2t) + \frac{1}{2}(\sin \pi t) \]
   \[ = (2t)P_r[X(t) = 2t] + \sin \pi tP_r[X(t) = \sin \pi t] \]

(b) \[ t = 0.25, \quad 2t = 1/2, \quad \sin \pi t = \sin \frac{\pi}{2} = 1 \]

Similarly:

\[ F_X(x; \frac{1}{2}) \]
\[ 1 = 2t \quad x = \frac{1}{2} \]
\[ F_X(x; 1) \]
\[ 1 = 2t \quad x = 1 \]
\[ 2 = 2t @ t = 1 \]
Final Examination

name _______________________
score ____________

1. Prob 5-3
2. Prob 5-14
3. Prob 8-23
4. Prob 8-24
5. Prob 9-1

6. A discrete stochastic process, \( x(n) \), is normalized to \( y(n) = A x(n) \) before an (fixed point) D conversion. We would like to have \( |y(n)| \leq 1 \) to avoid clipping. Assuming that \( x(n) \) is zero mean and \( \text{var} x(n) = \sigma_x^2 \) is known, find \( A \) so that

\[
\Pr [ |y(n)| \geq 1 ] \leq \frac{1}{16}.
\]

Instructions:
1. No outside human help.
2. All problems are equally weighted.
3. In take exams, neatness counts.
4. Mail to:
   Dr. Robert Marks
   16515 Ashworth Ave. N.
   Seattle WA 98133
   Postmarked no later than June 9, 1986

5. Use this sheet as a cover. Staple your work together in order.
6. Sign the following:

   "All outside references that I have used (human or other) are listed on the back side of this sheet."

\[ X \] ________________  
[sign]  \[ \text{date} \]
Name ____________________________  Score ____________

Instructions:
2. Two sheets of notes (stapled) and a calculator are okay.
3. Test time: 2:20 to 4:30 pm sharp.
4. All problems are equally weighted.
5. Do all of your work in this test booklet.
6. After the test is graded on campus students can pick up their test and grades at the EE main office. (No grades can be given over the phone). The students will have their tests returned as usual.

"... of making many books there is no end; and much study is a weariness of the flesh"

Ecc 12:12
-1

1.

(a) Find \( f_{Z}(z) \) given \( f_{Z|Y}(x,y) \) when

\[
Z = XY.
\]

(b) Apply your results in (a) when

\[
f_{Z|Y}(x,y) = \begin{cases} \frac{x}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}} & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}
\]
Let $\mathbf{X}$ denote an $N$ dimensional vector of iid random variables with mean $\mu$ and variance $\sigma^2$. Let $\mathbf{Y} = A \mathbf{X}$ where $A$ is some given $N \times N$ matrix; i.e.

$$Y_m = \sum_{n=1}^{N} a_{mn} X_n \quad ; \quad 1 \leq m \leq N$$

For large $N$, estimate the first order density of $Y_m$, as well as its mean and expected value.
The R.V.'s, \( \{X_n \mid 1 \leq n \leq 5 \} \) are iid and uniform on the interval \((-1, 1)\).
Compute:
\[
E \left[ (X_1 + X_2)^3 X_3 + (X_4 + X_5^2)^2 \right]
\]
The joint probability density function:

\[ f_{X,Y}(x,y) = 8 \gamma^2 e^{-2y} e^{-2x\gamma} U(x) U(y) \]

has a marginal density:

\[ f_Y(y) = 4 \gamma e^{-2y} U(y) \]

Given that \( \gamma = 1/2 \), what is a good estimate of \( X \)? [\( U(\cdot) = \text{unit step} \)]
The stochastic process \( X(t) \) is defined by:
\[
X(t) = 1 + t e^{-at}
\]
where the random variable \( \alpha \) is uniformly distributed on \((0, 1)\).
Consider the random variable
\[
Z = \int_0^Y X(t) \, dt
\]
Given that \( Y \) is also uniform on \((0, 1)\) and that \( Y \) and \( X \) are independent,
compute:
\[
\mathbb{E}[Z] = \mathbb{E}[Z]
\]
Hector Gleason manufactures fixed frequency oscillators. Two thirds of his units work. When turned on, they respond $\sin(\omega t) U(t)$ where $\omega$ is the fixed frequency. The other one third fizzles according to $e^{-\alpha t} U(t)$ where $\alpha$ is always the same. Let $X(t)$ be the waveform we obtain from a Gleason oscillator.

(a) What is the first order density, 
$$f_{X(t)}(x; t) = ?$$

(b) What is the correlation function:
$$R_X(t_1, t_2) = ?$$
1. (a) \( Z = XY \)

\[ F_Z(z) = P_r[Z \leq z] = P_r[XY \leq z] = P_r[Y \leq \frac{z}{x}, x \geq 0 \text{ or } Y \geq \frac{z}{x}, x < 0] \]

\[
F_Z(z) = \int_{x=0}^{\infty} \int_{y=0}^{\infty} f_{X,Y}(x, y) \, dx \, dy
\]

\[
f_Z(z) = \int_{x=0}^{\infty} \frac{1}{x} f_{X,Y}(x, \frac{z}{x}) \, dx + \int_{x=-\infty}^{0} (-1) f_{X,Y}(x, \frac{z}{x}) \, dx
\]

(b) \( f_{X,Y}(x, y) = \frac{x}{\sqrt{2\pi}} e^{-\frac{(xy)^2}{2}} \); \( 1 \leq x \leq 2 \)

\[
f_Z(z) = \int_{1}^{2} \frac{1}{x} \sqrt{\frac{x}{2\pi}} e^{-\frac{z^2}{2x}} \, dx
\]

\[= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = N(0,1) \]

\[ n_m = \frac{\sum_{n=1}^{N_m} a_{mn} X_n^2}{\sqrt{\sum_{n=1}^{N_m} a_{mn} \theta_{mn}}} \]

In general, will be approximately normal:

\[ f_{X_m}(y) \sim N(\mu_m, \sigma_m^2) \]

Thus

\[ E[X_{m+1}] = E[X_m] + \frac{1}{2} X_{m+1} \}

\[ E[X_{m+1}^3] = E[X_m^3] + \frac{1}{2} E[X_m^3] \}

\[ E[X_{m+1}^4] = E[X_m^4] + \frac{1}{2} E[X_m^4] \]

\[ E[X_{m+1}^5] = E[X_m^5] + \frac{1}{2} E[X_m^5] \]

\[ E[X_{m+1}^6] = E[X_m^6] + \frac{1}{2} E[X_m^6] \]
Solution

Use minimum \( \text{MSE} = \mathbb{E} = \int_{\mathbb{R}} x f^2(x) \, dx \).

\[ \mathbb{E} = \int \int f(x,y) \, dx \, dy \]

Integrate by parts:

\[ \int_{\mathbb{R}} x e^{-xy} \, dx = \int_{\mathbb{R}} y e^{-xy} \, dy \]

\[ x \int_{\mathbb{R}} e^{-xy} \, dx = y \int_{\mathbb{R}} e^{-xy} \, dy \]

\[ \mathbb{E} = \int_{\mathbb{R}} x f(x,y) \, dy \]

\[ \mathbb{E} = \int_{\mathbb{R}} y f(x,y) \, dx \]

\[ \mathbb{E} = \int_{\mathbb{R}} \left[ \int e^{-xy} \, dy \right] x \, dx \]

\[ \mathbb{E} = \int_{\mathbb{R}} \left[ \int e^{-xy} \, dy \right] x \, dx \]

\[ \mathbb{E} = 2Y \int_0^\infty e^{-2xy} \, dy = 1 \]

\[ \mathbb{E} = 2Y \int_0^\infty e^{-2xy} \, dy = 1 \]

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\[ \mathbb{E} = 2Y \int_0^\infty e^{-2xy} \, dy = 1 \]
5. \[ E Z = E \int_0^\gamma X(t) \, dt \]
\[ = E \left[ E \int_0^\gamma X(t) \, dt \mid Y = y \right] \]
\[ = E \left[ \int_0^\gamma E[X(t)] \, dt \mid Y = y \right] \]
\[ E X(t) = E \left[ 1 + t e^{-\alpha t} \right] \]
\[ = 1 + t \int_0^1 e^{-\alpha \tau} \, d\alpha = 1 - e^{-\alpha t} \bigg|_0^1 \]
\[ = 1 + [1 - e^{-t}] = e^{-t} \]
Thus:
\[ E Z = E \int_0^\gamma e^{-t} \, dt \mid Y = y \]
\[ = E 1 - e^{-Y} \]
\[ = 1 - \int_0^1 e^{-\gamma y} \, dy \]
\[ = 1 - [1 - e^{-1}] = e^{-1} \approx N_z \]

6. (a) \[ f_{X(t)}(x; t) = \frac{3}{2} \delta(x - \sin \omega t \, U(t)) + \frac{1}{3} \delta(x - e^{-\alpha t} \, U(t)) \]
(b) \[ R_{X}(t_1, t_2) = E \left[ X(t_1) \, X(t_2) \right] \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{12}(x_1, x_2) \, x_1 \, x_2 \, dx_1 \, dx_2 \]
where \( X(t_1) = x_1 \), \( X(t_2) = x_2 \) and
\[ f_{12}(x_1, x_2) = f_{X(t_1)X(t_2)}(x_1, x_2; t_1, t_2) \]
Now:
\[ f_{12}(x_1, x_2) = f_{12}(x_1, x_2 \mid X = \sin \omega t \, U(t)) \, P_r[X = \sin \omega t \, U(t)] \]
\[ + f_{12}(x_1, x_2 \mid X = e^{-\alpha t} \, U(t)) \, P_r[X = e^{-\alpha t} \, U(t)] \]
\[ = \frac{2}{3} \delta(x_1 - \sin \omega t \, U(t_1)) \delta(x_2 - \sin \omega t_2 \, U(t_2)) + \frac{1}{3} \delta(x_1 - e^{-\alpha t_1} \, U(t_1)) \delta(x_2 - e^{-\alpha t_2} \, U(t_2)) \]
and:
\[ R(t_1, t_2) = \left[ \frac{2}{3} \sin(\omega t_1) \sin(\omega t_2) + \frac{1}{3} e^{-\alpha(t_1 + t_2)} \right] U(t_1) \, U(t_2) \]
**Instruction:**
1. Test time: 2:20 to 4:20 PM  
   Mon, July 28
2. Closed Book \( \frac{1}{2} \) Notes.  
   One Sheet of Notes and Calculator OK.
3. Do all your work in the test booklet.

**Information:**
- All problems are worth 25 pts.

**Hints:**
\[
\begin{align*}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \cosh x + \sinh x\\
\cosh x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} ; \quad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\\
\text{sech } x &= 1 / \cosh x ; \quad \tanh x = \frac{\sinh x}{\cosh x}\\
\frac{d}{dx} \text{sech } x &= -\text{sech } x \cdot \tanh x\\
\frac{d}{dx} \tanh x &= \text{sech }^2 x\\
2\cosh x &= e^x + e^{-x}\\
2\sinh x &= e^x - e^{-x}\\
\text{sech } (0) &= \cosh (0) = 1\\
\sinh (0) &= \tanh (0) = 0\\
\int_{-\infty}^{\infty} b \text{sech} (\pi bx) e^{i\omega x} \, dx &= \text{sech} (\omega / 2b)
\end{align*}
\]
A fair coin (p=q=\(\frac{1}{2}\)) is flipped 1000 times. Compute the probability that 500 were heads.
$X$ is a Poisson random variable with parameter $\lambda$. What is the probability that $X$ is even? Is it greater than $\frac{1}{2}$?
Bill eats only chili-dogs and olive pizzas. Chili-dogs give him heartburn 10% of the time. The olive pizzas are worse. They give him heartburn 20% of the time. Bill eats twice as many chili dogs as pizzas. Bill has heartburn. What is the probability it was caused by a chili-dog?
The random variable $X$ has a pdf $f_X(x) = A x^a e^{-x/b}$ where $a$, $b > 0$.

(a) Find $A$.

(b) Compute $E[X]$ and $\text{var} X$.

(Do not ask proctor for $\int_0^\infty x e^{-x} dx$, $\int_0^\infty x^2 e^{-x} dx$. Rather, see the "Hints".)
A random variable $X$ is uniform on the interval $(0,1)$. Let $Y = g(X)$.

Sketch $f_Y(y)$. 

$g(x)$

$X$ $1$ $\frac{3}{4}$ $\frac{1}{2}$ $\frac{1}{4}$ 

$\frac{1}{2}$
The random variable, $X$, is the total time a lightbulb is functional given that it was turned on at time $0$. Assume 

$$f_X(x) = \alpha e^{-\alpha x} \mathcal{U}(x)$$

Suppose that the bulb worked to time $t$. Find $f_X(x | X > t)$ and note the lightbulb is as good as new.
EE505 Final
Part 1
Tues, 8-14-84
noon to 1 P.M.

1. Casey, the baseball player, has a batting average of 0.300 (i.e., $p = 0.3$ in a Bernoulli trial). Estimate the probability he gets over 850 hits (successes) in his next 3000 at bats (trials).
Let \( A = \frac{1}{N} \sum_{n=1}^{N} X_n \) where the \( X_n \)'s are iid:

\[
 f_{X_n}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2} ; n = 1, 2, \ldots, N
\]

Recall \( \Phi_{X_n}(\omega) = \exp[-\alpha |\omega|] \). Compute \( f_A(x) \)
Define $A = \frac{1}{N} \sum_{n=1}^{N} X_n$ where the $X_n$'s are iid with unit variance and zero mean.

Compute a lower bound on the probability that $A$ lies between $-a$ and $a$ where $a > 0$ is specified and $N$ is not large enough to apply the central limit theorem.
Casey, the baseball player, has a batting average of 0.300 (i.e., $p=0.3$ in a Bernoulli trial). Estimate the probability he gets over 1850 hits (successes) in his next 3000 at bats (trials).

**Solution:** Can use central limit theorem:

- $n=3000$, $p=0.3$, $q=0.7$
- $N=np=900$, $\sigma^2=npq=630$

$$Pr[k \leq 850] = G\left(\frac{850-900}{\sqrt{630}}\right)$$

$$= G[-1.99] = 0.0233 \quad \text{(From table)}$$

$$\therefore Pr[k > 850] = 0.9767 \approx 98\%$$
Let $A = \frac{1}{N} \sum_{n=1}^{N} X_n$ where the $X_n$'s are iid:

$$f_{X_n}(x) = \frac{\alpha/n}{\alpha^2 + x^2}; \quad n = 1, 2, \ldots, N$$

Recall $\Phi_{X_n}(\omega) = \exp[-\alpha |\omega|]$. Compute $f_A(x)$

**Solution**

$$\Phi_A(\omega) = E[e^{i\omega A}]$$

$$= E[e^{i\omega \frac{1}{N} \sum_{n=1}^{N} X_n}]$$

$$= \prod_{n=1}^{N} E[e^{i\omega X_n/N}]; \text{ since iid}$$

$$= \prod_{n=1}^{N} e^{-\alpha |\omega|/N}$$

$$= [e^{-\alpha |\omega|/N}]^N = e^{-\alpha |\omega|}$$

$$\Rightarrow f_A(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$$

Same as $f_{X}(x)$!

(no central limit theorem here!)
Define $A = \frac{1}{N} \sum_{n=1}^{N} X_n$, where the $X_n$'s are iid with unit variance and zero mean. Compute a lower bound on the probability that $A$ lies between $-a$ and $a$ where $a > 0$ is specified and $N$ is not large enough to apply the central limit theorem.

\[ \mathbb{E}[A] = 0, \quad \text{var} \ A = \frac{1}{N} = \sigma_A^2 \]

Use Chebychev inequality:

\[ P_r[|A| < k\sigma_A] \geq 1 - \frac{1}{k^2} \]

Set $k\sigma = a$

\[ P_r[|A| < a] \geq 1 - \left( \frac{\sigma_a}{a} \right)^2 = 1 - \frac{1}{N\sigma_a^2} \]

Note:

\[ P_r[|A| < \varepsilon] \xrightarrow{N \to \infty} 1 \quad \text{for all } \varepsilon \]

(A Law of Large Numbers)
EE505 Final
Part II
Thurs, 8-16-84
noon - 1 p.m.

1. Let \( Y \) be a gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \). Define the stochastic process \( X(t) = Y \) for all \( t \).

Compute:
(a) \( \mathcal{N}_X(t) \)
(b) \( R_X(t_1, t_2) \)
(c) \( C_X(t_1, t_2) \)
(d) \( \text{var} X(t) \)

(e) Is \( X(t) \) WSS?
\( X(t) \) is a WSS stochastic process with a first order density:

\[
 f_X(x) = e^{-x} U(x)
\]

and autocorrelation:

\[
 R_X(\gamma) = e^{-|\gamma|}
\]

What percentage of the time will \( X(t) \) exceed 1?
Recall the differentiation of the Poisson process:

\[ X(t) \xrightarrow{\frac{d}{dt}} \frac{d}{dt} \rightarrow Z(t) \]

we showed that \( E[Z(t)] = \lambda \) and that

\[ R_Z(\tau) = \lambda^2 + \lambda \delta(\tau) \]

Is \( Z(t) \) mean ergodic? Show your work.
Let $x(t)$ be stationary white noise:

$$R_x(\tau) = 9 \delta(\tau) \quad \mathbb{E}_X = 0$$

Let:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

(a) Compute the autocorrelation of $X(\omega)$.

(b) Is $X(\omega)$ WSS (widesense stationary)?
Fill in the blank with the best waveform. Use an answer only once. No guessing penalty.

(a) increasing mean
(b) increasing variance
(c) WSS
(d) constant mean
(e) constant variance
1. Let \( Y \) be a gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \). Define the stochastic process

\[
X(t) = Y \text{ for all } t.
\]

Compute:

(a) \( \mathbb{E}_x(t) \)

(b) \( R_x(t_1, t_2) \)

(c) \( C_x(t_1, t_2) \)

(d) \( \text{var}(X(t)) \)

(e) Is \( X(t) \) white?
$X(t)$ is a WSS stochastic process with a first order density:

$$f_X(x) = e^{-x} U(x)$$

and autocorrelation:

$$R_X(\tau) = e^{-|\tau|}$$

What percentage of the time will $X(t)$ exceed 1?

Solution

$$\Pr[X(t) > 1] = \int_{1}^{\infty} e^{-x} dx$$

$$= e^{-x} \bigg|_{1}^{\infty} = e^{-1} = 36.8\%$$

(independent of $R_X$)
Recall the differentiation of the Poisson process:

\[ \Xi(t) \quad \frac{d}{dt} \quad \Xi(t) \quad \Xi(t) \]

we showed that \( E[Z(t)] = \lambda \) and that

\[ R_Z(\tau) = \lambda^2 + \lambda \delta(\tau) \]

Is \( Z(t) \) mean ergodic? Show your work.

\[ \text{Solution} \quad \text{A sufficient condition for mean ergodicity is } Z(t) \text{ is wss} \]

and

\[ \int_{-\infty}^{\infty} |C_Z(\tau)| d\tau < \infty \]

Since

\[ C_Z(\tau) = R_Z(\tau) - \mathbb{E}^2_Z = \lambda \delta(\tau) \]

\[ \int_{-\infty}^{\infty} |C_Z(\tau)| d\tau = \lambda < \infty \]

yes, \( Z \) is mean ergodic.
Let \( X(t) \) be stationary white noise:

\[
R_x(\tau) = q \delta(\tau), \quad N_x = 0
\]

Let:

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt
\]

(a) Compute the autocorrelation of \( X(\omega) \).

(b) Is \( X(\omega) \) WSS (wide sense stationary)?

\[
S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega \tau} d\tau
\]

\[
= q
\]

\[
\Rightarrow R_x(\omega, \nu) = 2\pi q \delta(\omega - \nu)
\]

(b) Yes, \( R_x(\omega) = 2\pi q \delta(\omega) \)
Fill in the blank with the best waveform. Use an answer only once. No guessing penalty.

(a) increasing mean \( \overline{X(t)} \)
(b) increasing variance \( W(t) \)
(c) WSS \( Z(t) \)
(d) constant mean \( \overline{V(t)} \)
(e) constant variance \( U(t) \)
Problem 1: Multiple Choice (No guessing penalty)

A density function is equal to $Ax^2$ for $0 < x < 1$ and is otherwise zero.

Answers: (a) 0  (e) $\frac{3}{4}$  (i) $2^{1/3}$
(b) 1  (f) $\frac{3}{5}$  (j) $2^{-1/3}$
(c) 2  (g) $\frac{21}{80}$  (k) $\sqrt{3}$
(d) 3  (h) $\frac{3}{20}$  (l) None of the above

Questions: Use letters from answers above.

(i) $A = \ldots \ldots \ldots \ldots \ldots \ldots$
(ii) mean $= \ldots \ldots \ldots \ldots \ldots$
(iii) second moment $= \ldots \ldots \ldots$
(iv) variance $= \ldots \ldots \ldots$
(v) median $= \ldots \ldots \ldots$
(vi) mode $= \ldots \ldots \ldots$
(vii) range $= \ldots \ldots \ldots$

Note: An answer can be used more than once. Only your answers above will be graded.
Problem 2:

A Bernoulli trial with success probability $p$ is repeated until there is a failure. Let $X$ be the number of trials. Find $P[X=x]$. 
Problem 3:

θ is uniformly distributed on \((-\pi/2, \pi/2)\). \(Y = \sin \theta\). Find \(f_Y(y)\).

Hint: \(\frac{d}{dx} \arcsin x = \frac{1}{(1-x^2)^{\frac{1}{2}}}\).
Problem 4:

A joint density, \( f_{XY}(x,y) \), is equal to \( x + y \) on the unit square \( (0 \leq x \leq 1, 0 \leq y \leq 1) \) and is zero otherwise.

(a) Are \( X \) and \( Y \) independent?  

- Yes 
- No

(b) Compute \( P[X < \frac{1}{2} \mid Y = 1] \)
Problem 5:

X and Y are independent gamma random variables both with parameters b and c:

\[ f_X(x) = f_Y(x) = \frac{c^{b+1}}{\Gamma(b+1)} x^b \exp(-cx)U(x) \]

Let \( Z = X + Y \). Find \( f_Z(z) \)

Hint: \( \phi_X(\omega) = (1-j\omega)^{-b-1} \)
1. \[ f(x) = \begin{cases} \frac{1}{2}x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

(i) \[ \int_0^1 x^2 \, dx = 1 = A \cdot \frac{1}{2} x^3 \bigg|_0^1 = \frac{A}{3} \Rightarrow A = 3 \]

(ii) \[ \int_0^1 x^3 \, dx = \frac{1}{4} \Rightarrow \int_0^1 x^4 \, dx = \frac{3}{5} x^5 \bigg|_0^1 = \frac{3}{5} \Rightarrow e \]

(iii) \[ \int_0^1 x^4 \, dx = \frac{3}{5} \]

(v) \[ 3 \int_0^m x^2 \, dx = 3 \int_0^m x^2 \, dx = \frac{3}{2} \Rightarrow x^3 \bigg|_0^m = x^3 \bigg|_0^m = \frac{3}{5} \Rightarrow m = (\frac{1}{2})^{\frac{3}{5}} = 2^{-\frac{3}{5}} \Rightarrow f \]

(vi) Clearly, max is @ x = 1 \[ \Rightarrow b \]

(vii) Clearly, 1 \[ \Rightarrow b \]

2. \[ P_r [X = 1] = q \quad P_r [X = 4] = p^3 q \]

\[ P_r [X = 2] = pq \]

\[ P_r [X = 3] = p^2 q \quad P_r [X = x] = p^{x-1} q \leftrightarrow \text{geometric} \]

3. \[ Y = 3 \sin \Theta \Rightarrow \Theta = \sin^{-1}(Y) = \sin^{-1} \quad \text{strictly increasing} \]

\[ f_X(y) = \frac{d}{dy} f_{\Theta}(\sin^{-1}(y)) \]

\[ f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} ; & |\theta| < \pi/2 \\ 0 ; & \text{otherwise} \end{cases} \]

\[ -\pi/2 < \sin^{-1}Y < \pi/2 \Rightarrow -1 < Y < 1 \]

4. (a) NO! \( f_X(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \]

\[ f_Y(y) = \int_{\frac{y}{2}}^{\frac{1+y}{2}} x \, dx = \frac{x^2}{2} + y \times \frac{1}{2} = \frac{1}{2} + y \]

\[ \therefore f_X(x/y) = \frac{x+1}{\frac{1}{2}+y} \text{ on the unit square given} \]

\[ P[X < 1/2 | Y = 1] = \int_0^{1/2} x \frac{x+y}{\frac{1}{2}+y} \, dx = \frac{2}{3} \left( \frac{1}{3} + \frac{1}{2} \right) = \frac{2}{3} \cdot \frac{5}{12} = \frac{5}{18} \]

5. \[ \Phi_Z(w) = \Phi_X(w) \Phi_Y(w) = (1-jcw)(1-jcw) = (1-jcw)^2(2b+1) \]

\[ f_Z(z) = \frac{C^{2b+2}}{\Gamma(2b+2)} x^{2b+1} e^{-cx} U(x) \]
Solutions

1. \( f_X(x) = \begin{cases} \frac{1}{2}x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

(i) \( \int_0^1 A x^2 \, dx = 1 \Rightarrow A = \frac{1}{3} \)

(ii) \( 3 \int_0^1 x^3 \, dx = 3 \frac{1}{4} \Rightarrow e \)

(iii) \( 3 \int_0^1 x^4 \, dx = \frac{7}{3} \Rightarrow f \)

(iv) \( \sigma^2 = \frac{1}{3} - \frac{9}{16} = \frac{4\pi - 21}{90} \Rightarrow \frac{8}{\pi} \)

(v) \( 3 \int_0^m x^2 \, dx = 3 \int_0^1 x^2 \, dx = \frac{1}{2} \Rightarrow x^3 \bigg|_0^m = x^3 \bigg|_0^1 \)

\( m^3 = 1 - m^3 \Rightarrow 2m^3 = 1 \Rightarrow m = (\frac{1}{2})^{\frac{1}{3}} = 2^{-\frac{1}{3}} \Rightarrow \)

(vi) Clearly, max is @ \( x = 1 \)

(vii) Clearly, \( 1 \)

2. \( P_r[X = 1] = p \quad P_r[X = 4] = p^3 q \\
   P_r[X = 2] = pq \\
   P_r[X = 3] = p^2 q \\
   P_r[X = 4] = p^3 q \quad \text{geometric} \)

3. \( Y = \sin \theta \Rightarrow \theta = g^{-1}(y) = \arcsin y \leftarrow \text{strictly increasing on } (-\pi, \pi) \)

\( f_\theta(\theta) = \begin{cases} \frac{1}{\pi} \quad |\theta| < \pi/2 \\ 0 \quad \text{otherwise} \end{cases} \)

\( f_\theta(\theta) \begin{cases} \left(\frac{1}{\sqrt{1-y^2}} \right) ; \\ 0 \end{cases} \quad \text{otherwise} \)

\( -\pi/2 < \arcsin y < \pi/2 \Rightarrow -1 < y < 1 \)

4.(a) NO! (b) \( f_{x|y}(x|y) = f_{x,y}(x,y) / f_Y(y) \)

\( f_x(y) = \int_0^1 (x+y) \, dx = \frac{3}{2} + y \times 1 = \frac{3}{2} + y \)

\( \therefore f_{x|y}(x|y) = \frac{x+y}{\frac{3}{2} + y} \text{ on the unit square} \)

\( \rho[X < \frac{1}{2}, X = 1] = \int_{\frac{1}{2}}^1 \frac{x+y}{\frac{3}{2} + y} \, dx \bigg|_{y=0}^{\frac{1}{2}} = \frac{3}{2} \int_0^{\frac{1}{2}} (x+1) \, dx \)

\( = \frac{3}{2} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{3}{2} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} = \frac{1}{2} \)

5. \( \Phi_\mathbf{Z}(\omega) = \Phi_X(\omega) \Phi_\mathbf{Z}(\omega) = (1-jc\omega)^{-2b-2} = (1-jc\omega)^{-2b-2} \)

= characteristic function of gamma r.v. with parameters \( 2b+1 \).

\( \therefore f_\mathbf{Z}(\mathbf{z}) = \frac{C^{2b+2}}{\Gamma(2b+2)} \times 2b+1 e^{-cz} u(x) \)
1. One fair coin is flipped 2 times. Are the 2 events
   \[ A: \text{a head occurs on the first flip} \]
   \[ B: \text{a head occurs on the second flip} \]
   independent? **YES**

2. A fair coin is flipped 2 times. Let \( A \) be the event that a head occurs on the first flip and let \( B \) be the event that the same face does not occur on both flips. Are \( A \) and \( B \) independent? **YES**

3. An urn contains 4 balls numbered 1, 2, 3, 4, respectively. Two balls are drawn without replacement. Let \( A \) be the event that the first ball drawn has a 1 on it and let \( B \) be the event that the second ball has a 1 on it. Are \( A \) and \( B \) independent? **NO**

4. If the drawing is done with replacement in problem 3, are \( A \) and \( B \) independent? **YES**

5. A pair of dice is rolled 1 time. Let \( A \) be the event that the first die has a 1 on it, \( B \) the event that the second die has a 6 on it, and \( C \) the event that the sum is 7. Are \( A, B, \) and \( C \) independent? **NO**

6. A fair coin is flipped 3 times. Let \( A \) be the event that a head occurs on the first flip, let \( B \) be the event that at least 2 tails occur, and let \( C \) be the event that we get exactly 1 head or that we get tail, head, head in that order. Show that these 3 events satisfy equation 4 of Definition 2.7.3, but not equations 1, 2, or 3.

7. Prove that if \( A \) and \( B \) are independent, so are \( A \) and \( B \).

8. The probability that a certain basketball player scores on a free throw is .7. If in a game he gets 15 free throws, compute the probability that he makes them all. Compute the probability that he makes 14 of them. What assumptions have you made in deriving your answer? \((0.7)^{15}, (0.7)^{14}(0.3)\)

9. Three teams, \( A, B, \) and \( C \), enter a round-robin tournament. (Each team plays 2 games, 1 against each of the possible opponents. The winner of the tournament, if there is a winner, is the team winning both its games.) Assume that the game played is one in which a tie is not allowed. We assume the following probabilities:

   \[
   P(A \text{ beats } B) = .7 \\
   P(B \text{ beats } C) = .8 \\
   P(C \text{ beats } A) = .9.  
   \]

Compute the probability that team \( A \) wins the tournament; that team \( B \) wins the tournament. Compute the probability no one wins the tournament.

0.07, 0.24, 0.51

2.8.

1. A fair die is rolled until a 1 occurs. Compute the probability that:
   \( (a) \) 10 rolls are needed \( (\frac{5}{6})^9 (\frac{1}{6}) \)
   \( (b) \) less than 4 rolls are needed \( 6/11 \)
   \( (c) \) an odd number of rolls is needed \( 6/11 \)

2. A fair pair of dice is rolled until a 7 occurs (as the sum of the 2 numbers on the dice). Compute the probability that
   \( (a) \) 2 rolls are needed \( 5/36 \)
   \( (b) \) an even number of rolls is needed \( 5/11 \)

3. You fire a rifle at a target until you hit it. Assume the probability that you hit it is .9 for each shot and that the shots are independent. Compute the probability that:
   \( (a) \) it takes more than 2 shots \( 0.01 \)
   \( (b) \) the number of shots required is a multiple of 3 \( 1/11 \)

4. Hugh takes a written driver's license test repeatedly until he passes it. Assume the probability that he passes it any given time is .1 and that the tests are independent. Compute the probability that:
   \( (a) \) it takes him more than 4 attempts \( (0.9)^4 \)
   \( (b) \) it takes him more than 10 attempts \( (0.9)^{10} \)

5. A traffic light on a route you travel every day turns red every 4 minutes, stays red 1 minute and then turns green again (thus it is green 3 minutes, red 1, etc.), with the red part of the signal starting on the hour, every hour.

   \( (a) \) If you arrive at the light at a random instant between 7:55 a.m. and 8:05 a.m., what is the probability that you have to stop at the light? \( 3/10 \)
   \( (b) \) If you arrive at the light at a random instant between 7:54 a.m. and 8:04 a.m., what is the probability that you have to stop for the light? \( 2/10 \)

6. The plug on an electric clock with a sweep second hand is pulled at a random instant of time within a certain minute. What is the probability that the second hand is between the 4 and the 5? Between the 1 and the 2? Between the 1 and the 6?

\[
\frac{1}{12}, \frac{1}{12}, \frac{5}{12}
\]

7. A point is chosen at random between 0 and 1 on the x-axis in the \((x, y)\) plane. A circle centered at the origin is then drawn in the plane, with radius determined by the chosen point. Compute the probability that the area of the circle is less than \( \pi/2 \).  

\[
\frac{1}{2} \sqrt{\frac{3}{2}}
\]

8. A 12-inch ruler is broken into 2 pieces at a random point along its length. What is the probability that the longer piece is at least twice the length of the shorter piece? \( 2/3 \)
12. Given \( f_X(x) \) as probability density function:
\[
F_X(t) = \begin{cases} 
0 & ; t < 99 \\
\frac{t}{99} & ; 99 \leq t < 100 \\
1 & ; t \geq 100
\end{cases}
\]
Derive \( F_X(t) \).

13. \( Y \) is a continuous random variable with \( f_Y(y) = 2(1 - y) \), \( 0 < y < 1 \) otherwise.
Derive \( F_Y(t) \).

14. \( Z \) is a continuous random variable with probability density function
\[
f_Z(z) = 10e^{-10z}, \quad z > 0
\]
\[
f_Z(z) = 0, \quad \text{otherwise}
\]
Derive \( F_Z(t) \).

EXERCISE 3.3.

1. If
\[
P_X(x) = \begin{cases} 
\frac{1}{6} & ; x = 2, 4, 8, 16 \\
0 & , \text{otherwise}
\end{cases}
\]
compute:
(a) \( E[X] \) (b) \( E[X^2] \)

2. Suppose that \( f_X(x) = \frac{1}{6}, \quad -1 < x < 1 \), compute:
(a) \( E[X] \) (b) \( E[X^2] \)

3. Given
\[
f_X(x) = 2(1 - x), \quad 0 < x < 1
\]
compute:
(a) \( E[X] \) (b) \( E[X^2] \)

4. Show that \( E[X - \mu_X] = 0 \).

EXERCISE 3.5.

1. Let \( X \) be a random variable with distribution function \( F_X(t) \) and let \( Y = a + bX \) where \( b < 0 \). Derive the distribution function for \( Y \).

2. Suppose that \( b = 0 \) in problem 1 above. Derive the distribution function for \( Y \), defined as in that problem.

3. Given
\[
F_X(t) = \begin{cases} 
0 & ; t < -1 \\
\frac{t + 1}{2} & ; -1 \leq t \leq 1 \\
1 & ; t > 1
\end{cases}
\]
find the distribution function for \( Y = 15 + 2X \) and the density function for \( Y \).

4. Suppose that
\[
F_Y(t) = \begin{cases} 
0 & ; t < 0 \\
t^2 & ; 0 \leq t \leq 1 \\
1 & ; t > 1
\end{cases}
\]
and let \( Z = W - 1 \). Find \( F_Z(t) \) and \( f_Z(t) \).
\[
F_Z(t) = \begin{cases} 3(t+1)^2 & ; -1 \leq t \leq 0 \\
\end{cases}
\]
10. Let $U$.

2. If $X$.

3. Let $X$.

4. Let $X$.

5. Let $X$.


7. Let $X$.

8. Let $X$.

9. Let $X$.

10. Let $X$.

11. Let $X$.

EXERCISE 4.3.

1. It has been observed that cars pass a certain point on a rural road at the average rate of 3 per hour. Assume that the instants at which the cars pass are independent and let $X$ be the number that pass this point in a 30-minute interval. Compute $P(X = 0), P(X \geq 2)$. $0.2231, 0.4422$.

2. It has been observed empirically that deaths per hour, due to traffic accidents, occur at a rate of 8 per hour on long holiday weekends in the United States. Assuming that these deaths occur independently, compute the probability that a 1-hour period would pass with no deaths; that a 15-minute period would pass with no deaths; that 4 consecutive, nonoverlapping 15-minute periods would pass with no deaths. $0.0003, 0.1353, 0.0003$.

3. It has been observed that packages of Hamm's beer are removed from the shelf of a particular supermarket at a rate of 10 per hour during rush periods. What is the probability that at least 1 package is removed during the first 10 minutes of a rush period? What is the probability that at least 1 is removed from the shelf during each of 3 consecutive, nonoverlapping 10-minute intervals? $0.8111, 0.0003$.

4. At a certain manufacturing plant, accidents have been occurring at the rate of 1 every 2 months. Assuming that the accidents occur independently, what is the expected number of accidents per year? What is the standard deviation of the number of accidents per year? What is the probability of there being no accidents in a given month? $6, \sqrt{6}, 0.6065$.

5. Suppose that quarter-pound bars of butter are cut from larger slabs by a machine. We assume that the larger slabs are quite uniform in density; if the length of the bar is exactly 3\(\frac{1}{2}\) inches, then the bar will weigh \(\frac{1}{2}\) pound. Suppose that the true length $X$ of a bar cut by this machine is equally likely to lie in the interval from 3.35 inches to 3.45 inches. Assuming that the lengths of bars cut by this machine are independent, what is the probability that all 4 bars in a particular pound package of butter will weigh at least \(\frac{1}{2}\) pound? That exactly 3 will weigh at least \(\frac{1}{2}\) pound? $0.316, 0.422$.

6. $X$ is uniformly distributed on $(0, 2)$ and $Y$ is exponential with parameter $\lambda$. Find the value of $\lambda$ such that $P(X < 1) = P(Y < 1)$. $0.69$.

7. Calls arrive at a switchboard according to a Poisson process with parameter $\lambda = 5$ per hour. If we are at the switchboard, what is the probability that it is at least 15 minutes until the next call? That it is no more than 10 minutes? That it is exactly 5 minutes?

8. A newsboy is selling papers on a busy street. The papers he sells are events in a Poisson process with parameter $\lambda = 50$ per hour. If we have just purchased a paper from him, what is the probability that it will be at least 2 minutes until he sells another? If it is already 5 minutes since his last sale, what is the probability it will be at least 2 more minutes until his next sale? $0.1882, 0.1882$.

9. $X$ is uniform on $(-1, 3)$ and $Y$ is exponential with parameter $\lambda$. Find $\lambda$ such that $\sigma_X^2 = \sigma_Y^2$. $\frac{1}{2} \sqrt{3}$.

10. $X$ is geometric with parameter $p$ and $Y$ is exponential with parameter $\lambda$. Find $\lambda$ such that $P(X > 1) = P(Y > 1)$. $-\ln (1 - p)$.

11. We are given a Poisson process with parameter $\lambda$. We begin observing the process at time zero; let $S$ be the time until the second event occurs. Derive the probability density function for $S$. $\left(1 - 2t \right)^{-1} \sqrt{2}$.
EXERCISE 4.5.

1. Assume that the time $X$ required for a distance runner to run a mile is a normal random variable with parameters $\mu = 4$ minutes, 1 second and $\sigma = 2$ seconds. What is the probability that this athlete will run the mile in less than 4 minutes? In more than 3 minutes, 55 seconds?

0.3085, 0.9773, 0.0227, 0.0227

2. The length $X$ of an adult rock cod caught in Monterey Bay is a normal random variable with parameters $\mu = 16$ inches and $\sigma = 1$ inch. If you catch one of these fish, what is the probability that it will be at least 14 inches long? That it will be no more than 17 inches long? That its length will be between 12 inches and 15 inches? 0.774, 0.8413, 0.1587

3. If $Z$ is a standard normal random variable and we define $U = |Z|$, then $U$ is called the folded standard normal variable. Express $F_U(t)$ in terms of $F_Z(t)$.

4. Suppose that we are given a target with a vertical straight line drawn through its center. Let us assume that if we throw a dart at this target and measure the distance $Z$ between the point we hit and the center line, then $Z$ is a standard normal random variable (if the dart lands right of the center line the measurement is positive, if it lands to the left of the center line the measurement is negative). Then, the distance from the point we hit to the center line is $|Z| = U$, the folded normal random variable defined in problem 3. Compute $P(U > 1)$ and $P(U < 1)$. 0.3147, 0.3830

# 7. Show that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \, dx = 1$$

for any $\mu$ and for $\sigma > 0$. (Hint: If

$$A = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \, dx,$$

then

$$A^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma^2} e^{-(x-\mu)^2/(2\sigma^2) - (y-\mu)^2/(2\sigma^2)} \, dx \, dy;$$

let $u = (x - \mu)/\sigma$, $v = (y - \mu)/\sigma$, and transform to polar coordinates to show $A^2 = 1$ which implies $A = 1$.)

EXERCISE 5.5.

1. If $X$ is uniformly distributed on the interval $(0, 1)$, compare $P(X - \mu < k \sigma X)$ with the values given by the Chebyshev inequality for $k = 1$, $\frac{1}{2}$, and 2.

2. For any value of $k \geq 1$, we can define a discrete random variable $Y$ to have probability function

$P_Y(y) = \frac{k^2 - 1}{k^2 - \frac{1}{2} - 2k};
\begin{cases}
  1, & y = 0, \\
  \frac{1}{2k^2}, & y = -k, k, \\
  0, & \text{otherwise}.
\end{cases}$

Compute $P_Y$ and compare the exact probability $P(X - \mu < k \sigma X)$ with the bound given by Chebyshev's inequality.
1. Three dice are rolled. Consider the following events:

A: The outcome on the blue die is odd
B: The outcome on the red die is even
C: The outcome on the yellow die is one
D: The sum of the red and yellow dice is even
E: The sum of the red and yellow dice is four
F: The sum of the red and yellow dice is five
G: The sum of the blue and yellow dice is three
H: The sum of the blue and yellow dice is two
I: The sum of all three dice is three
J: You will pass the test

Using the notation:

m = mutually exclusive
i = independent
s = one event is a subset of the other
n = none of the above

classify the following event pairs:

A and B
A and C
B and D
C and D
D and E
C and H
C and G
C and I
H and I
A and J

\[ \text{since } P_r[D/B] = P_r[D] = \frac{1}{2} \]
Compute the distribution function, $F_X(x)$, for a Rayleigh random variable with parameter $\alpha$.

$$f_X(x) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} U(x)$$

$$F_X(x) = \frac{1}{\alpha^2} \int_0^x \frac{e^{-\frac{\xi^2}{2\alpha^2}}}{\xi} d\xi$$

$$= \frac{1}{\alpha^2} \left[ e^{-\frac{x^2}{2\alpha^2}} \right]_0^x$$

$$= \frac{1}{\alpha^2} \left[ 1 - e^{-\frac{x^2}{2\alpha^2}} \right] U(x)$$

Since $F_X(\infty) = 1$

$$F_X(x) = \left[ 1 - e^{-\frac{x^2}{2\alpha^2}} \right] U(x)$$
Let $X$ be a Laplace random variable. Define $Y = U(X)$ where $U()$ is the unit step. Find the density function for $Y$.

Clearly, if $X \geq 0$, $Y = 1$
if $X < 0$, $Y = 0$

Thus, half the mass goes to zero and half to one:

\[ f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq y < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases} \]

or

\[ f_Y(y) = \frac{1}{2} \delta(y) + \frac{1}{2} \delta(y - 1) \]
CONSIDER THE TRUNCATED CAUCHY DENSITY:

\[ f_X(x) = \begin{cases} \frac{A}{(x^2 + 1)^{1/2}} & ; |x| \leq 1 \\ 0 & ; \text{otherwise} \end{cases} \]

(a) Compute \( A \)
(b) Compute \( E(X) \)
(c) Compute \( \text{var}(X) \)
(d) Compute the conditional density, \( f_X(x \mid x > 0) \).

\[ \int_{-\infty}^{\infty} f_X(x) \, dx = 1 = A \int_{-1}^{1} \frac{dx}{x^2 + 1} = A \tan^{-1} x \bigg|_{-1} = A \left[ \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right] = \frac{\pi A}{2} \Rightarrow A = \frac{2}{\pi} \]

\( b. \quad E[X] = 0 \) since \( f_X \) is even

\( c. \quad \text{var} X = E[X^2] = \frac{2}{\pi} \int_{-1}^{1} \frac{x^2 \, dx}{x^2 + 1} = \frac{2}{\pi} \left[ \frac{x}{2} - \tan^{-1} x \right] \bigg|_{-1}^{1} = \frac{2}{\pi} \left[ (1 - \tan^{-1} 1) - (1 - \tan^{-1} 1) \right] = \frac{2}{\pi} \left[ (1 - \frac{\pi}{4}) - (1 + \frac{\pi}{4}) \right] = \frac{2}{\pi} \left[ 1 - \frac{\pi}{4} + 1 - \frac{\pi}{4} \right] = \frac{2}{\pi} \left[ 2 - \frac{\pi}{2} \right] = \frac{4}{\pi} - 1 \]

\( d. \quad f_X(x) \Rightarrow \quad f_X(x \mid x > 0) = \frac{4/\pi}{x^2 + 1} ; \quad 0 \leq x \leq 1 \quad \text{o.w.} \)

\[ f_X(x \mid x > 0) = \begin{cases} \frac{4/\pi}{x^2 + 1} & ; \quad 0 \leq x \leq 1 \\ 0 & ; \text{otherwise} \end{cases} \]
Four major prizes are to be given away in a state lottery. Four drawings are made without replacement from all entries. Thus, one entry can at most win one prize. You enter the lottery six times. The total number of entries (including yourself) is 10,000. What is the probability you will win at least one of the major prizes? Give a number for your answer.

\[
P[\text{win}] = 1 - P[\text{loose}]
\]

\[
P[\text{loose}] = \frac{9994}{10,000} \cdot \frac{9993}{9999} \cdot \frac{9992}{9998} \cdot \frac{9991}{9997}
\]

\[
= 0.99760
\]

\[
\Rightarrow P_r[\text{win}] = 0.00240
\]
You perform a Bernoulli trial. The chance of success is p. You perform the trial until you get a failure. Let \( N \) be the random variable equal to the number of trials performed.

(a) What is \( \Pr(N=m) \) for some given \( m \)?
(b) Find the pdf, \( f_N(x) \).
(c) Compute \( E(N) \).

\[
\begin{align*}
(a) \quad & P_r[N=1] = p \\
& P_r[N=2] = p^2q \\
& P_r[N=3] = p^3q \\
& \quad \vdots \\
& P_r[N=m] = p^{m-1}q \\
(b) \quad & f_N(x) = \sum_{k=1}^{\infty} p^{k-1}q \delta(x-k) \\
(c) \quad & E[X] = \sum_{k=1}^{\infty} kp^{k-1} \\
& \quad = \frac{1}{(1-p)^2} \left[ \frac{1}{p} \right] \quad \Rightarrow \quad \text{Given as} \quad \frac{E[X]}{q} = \frac{1}{(1-p)^2} = \sum_{k=1}^{\infty} kp^{k-1} \quad \Rightarrow \quad \text{"Hint"} \\
& E[X] = \frac{1}{(1-p)^2} = \frac{1}{q} \\
\text{Note: For coin, } p = 2
\end{align*}
\]
A random variable has unit variance and zero mean. You wish to set a threshold, $T$, so that the probability of the magnitude of the random variable exceeding $T$ is no greater than one chance in a hundred. What is a good value for $T$? Justify your choice.

Chebyshev's Inequality for zero mean, $\sigma = 1$:

$$P_r[|X| \geq k] \leq \frac{1}{k^2} = \frac{1}{100}$$

$k = 10$

Thus set $T = 10$
1. Three dice are rolled. Consider the following events:

A: The outcome on the blue die is odd
B: The outcome on the red die is even
C: The outcome on the yellow die is one
D: The sum of the red and yellow dice is even
E: The sum of the red and yellow dice is four
F: The sum of the red and yellow dice is five
G: The sum of the blue and yellow dice is three
H: The sum of the blue and yellow dice is two
I: The sum of all three dice is three
J: You will pass the test

Using the notation:

m = mutually exclusive
i = independent
s = one event is a subset of the other
n = none of the above

classify the following event pairs:

A and B
A and C
B and D
C and D
D and E
C and H
C and G
C and I
H and I
A and J

Helpful hints to use elsewhere on this test:

\[ \frac{d}{dx} e^{x^2} = 2x e^{x^2} \]

\[ \int \frac{dx}{x^2 + 1} = \arctan x \]

\[ \sum_{k=0}^{\infty} a^k = (1 - a)^{-1}; |a| < 1 \]

\[ \sum_{k=1}^{\infty} k a^{k-1} = (a - 1)^{-2}; |a| < 1 \]
Compute the distribution function, \( F_X(x) \), for a Rayleigh random variable with parameter \( \sigma \).
Let $X$ be a Laplace random variable. Define $Y = U(X)$ where $U()$ is the unit step. Find the density function for $Y$. 
CONSIDER THE TRUNCATED CAUCHY DENSITY:

\[ f_X(x) = \begin{cases} A(x^2 + 1)^{-1} & ; |x| \leq 1 \\ 0 & ; \text{otherwise} \end{cases} \]

(a) Compute A
(b) Compute \( E(X) \)
(c) Compute \( \text{var}(X) \)
(d) Compute the conditional density, \( f_X(x/ X > 0) \).
Four major prizes are to be given away in a state lottery. Four drawings are made without replacement from all entries. Thus, one entry can at most win one prize. You enter the lottery six times. The total number of entries (including yourself) is 10,000. What is the probability you will win at least one of the major prizes? Give a number for your answer.
You perform a Bernoulli trial. The chance of success is $p$. You perform the trial until you get a failure. Let $N$ be the random variable equal to the number of trials performed.

(a) What is $\Pr(N=m)$ for some given $m$?
(b) Find the pdf, $f_N(x)$.
(c) Compute $E(N)$. 

A random variable has unit variance and zero mean. You wish to set a threshold, $T$, so that the probability of the magnitude of the random variable exceeding $T$ is no greater than one chance in a hundred. What is a good value for $T$? Justify your choice.
1. A random variable $X$ has a characteristic function:

$$\phi_X(t) = \frac{1}{1 + 2\sigma \cos(t/2) + \sigma^2}$$

where $\sigma$ is a given parameter. Compute:

(a) $\phi_X(0) = 1 \Rightarrow A = 1$

(b) $d\phi_X(t) = \phi_X(t) \left( -2\sigma \sin(t/2) + \frac{t^2}{2} \right)$

(c) $d^2\phi_X(t) = \phi_X(t) \left( -2\sigma^2 \cos(t/2) + \frac{t^4}{4} + \sigma^2 \right)$

$$\phi_X(t) = \begin{cases} 0, & t = 0 \\ 2\sigma^2 - 2\sigma^2 \cos(t/2) + \frac{t^2}{2}, & t \neq 0 \end{cases}$$
Sometimes the expected value of a random variable is not such a good estimate. For example, let $X$ be a Poisson random variable with parameter $\lambda = 1$. Let $Y = (-1)^X$.

Compute $E(Y)$ and comment.

$$E[Y] = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-\lambda} \lambda^k}{k!}$$

$\lambda = 1$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{(e\lambda)^k}{k!} = e^{-1} e^{-\lambda}$$

$$= e^{-2}$$

$$= 0.135$$

**COMMENT:** $Y$ is always $\pm 1$.
Let \(X_1, X_2, \ldots, X_n\) be i.i.d. random variables with a gamma density. Define

\[ f_X(x) = \frac{1}{\Gamma(k)} \left( \frac{x}{\theta} \right)^{k-1} e^{-x/\theta} \quad \text{for} \quad x > 0, \]

where \(k, \theta > 0\). Let the density function for the average, \(f_{\bar{X}}(x)\).

From the Fourier transform, recalling the theory:

\[ \Phi_{\bar{X}}(w) = \left( \frac{c}{c - jw} \right)^n \]

\[ f_{\bar{X}}(x) = \frac{N}{(\theta + N\lambda)^n} \left( \frac{\lambda}{\theta + N\lambda} \right) \]

\[ \Phi_{\bar{X}}(w) = \left( \frac{e^{-j\lambda w}}{e^{-j\lambda w} - 1} \right)^n \]

\[ \bar{X} = E[X_1 + \cdots + X_n] \]
Let $P$ and $Q$ denote independent random variables, both uniformly distributed on the interval from zero to unity. Define the process:

$$X(t) = P \cdot e^{-Qt} \cdot U(t)$$

Find:
(a) $E[X(t)]$
(b) $R(t, t_2) = E[X(t)X(t_2)]$
(c) $\text{var} X(t)$

(a) $E[X(t)] = E\left[ P \cdot e^{-Qt} \cdot U(t) \right]$

$$= E[P] \cdot E\left[ e^{-Qt} \cdot U(t) \right] ; E[P] = \frac{1}{2}$$

$$E[e^{-Qt}] = \int_0^1 e^{-Qt} \, dq = -\frac{1}{t} e^{-Qt}\big|_0^1 = \frac{1-e^{-t}}{t}$$

$$\Rightarrow E[X(t)] = \frac{1-e^{-t}}{2t} \cdot U(t)$$

(b) $R(t, t_2) = E[X(t)X(t_2)] = E\left[ P^2 \cdot e^{-Q(t+t_2)} \right] U(t) U(t_2)$

$$E[P^2] = \int_0^1 P^2 \, dp = \frac{1}{2}$$

$$E[e^{-Q(t+t_2)}] = \frac{1-e^{-(t+t_2)}}{t+t_2}$$

$$\Rightarrow R(t, t_2) = \frac{1-e^{-(t+t_2)}}{2(t+t_2)} \cdot U(t) U(t_2)$$

(c) $E[X^2(t)] = R(t, t) = \frac{1-e^{-2t}}{6t} \cdot U(t)$

$$\text{var} X(t) = E[X^2] - E(X)^2 = \left( \frac{1-e^{-2t}}{6t} - \left( \frac{1-e^{-t}}{2t} \right)^2 \right) U(t)$$
We draw $N$ iid samples from a shifted Laplacian random variable with mean $\lambda$ and variance $\sigma^2$. Give an approximation of the density function for the average of these numbers, $N > 1$.

Control limit theorem:

\[ \bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n \]

\[ E[\bar{X}] = E[X] = \lambda \]

\[ \text{Var} \bar{X} = \frac{\sigma^2}{N} \]

\[ \Rightarrow \bar{X} \text{ is normal (mean} = \lambda, \text{ variance} = \frac{\sigma^2}{N}) \]

\[ \frac{\bar{X} - \lambda}{\sqrt{\frac{\sigma^2}{N}}} \sim \mathcal{N} \]
$X(t)$ is a stationary random process with mean $\mu$ and autocorrelation

$$R(\tau) = \mu^2 \exp(-\alpha |\tau|)$$

where $\alpha$ is a specified parameter. What percentage of the time can we expect $X(t)$ to lie below a given threshold, $T$?

$$Pr[X(t) \leq T] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \text{var}}} e^{-\frac{(x-\mu)^2}{2\text{var}}} \, dx$$

Let $y = \frac{x-\mu}{\sqrt{\text{var}}}$

$$Pr[X(t) \leq T] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \text{var}}} e^{-\frac{y^2}{2\text{var}}} \, dy$$

$$= \frac{1}{2} + \text{erf} \left( \frac{T-\mu}{\sqrt{\text{var}}} \right)$$

$$= \frac{1}{2} + \text{erf} \left( \frac{T-\mu}{\sqrt{\mu^2 - \sigma^2}} \right)$$
In our take-home problem last week, we found that the joint density

\[ f(x,y) = 8y e^{-2y} e^{-2y} u(x) u(y) \]

had a marginal density

\[ f_Y(y) = 4y e^{-2y} u(y) \]

Given that \( I \) what is a good estimate of \( X? \)

**Minimum MSE is**

\[ f_{X|Y}(x|y) = \frac{f_{x,y}(x,y)}{f_Y(y)} = \frac{3xe^{-2y}}{4y} \]

\[ E[X|Y] = 2y \int_0^\infty xe^{-2y} d\lambda(x) \]

\[ = 2y \left[ \frac{1}{2} y \right] = \frac{1}{2} y \]

Choose \( \lambda = \frac{1}{4} \)
useful relationships:

Beta density: \( f_X(x) = \frac{\Gamma(b+1)\Gamma(c+1)}{\Gamma(b+c+1)} x^{b-1}(1-x)^{c-1} \) on \( 0 \leq x \leq 1 \)

Normal Density: \( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right) \) on \( -\infty < x < \infty \)

Poisson density: \( f_X(x) = e^{-\lambda} \frac{\lambda^x}{x!} \) on \( 0 < x < \infty \)

Gaussian: \( F_X(x) = \frac{1}{2} + \text{erf}\left(\frac{x-\mu}{\sigma}\right) \)

\( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du \)

\( \Gamma(b,c) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} ; \Gamma(n+1) = n! \)

1. A random variable has a probability density function:

\[
f_X(x) = \begin{cases} 
A/x^2 & ; 1 \leq x \leq 2 \\
0 & ; \text{otherwise}
\end{cases}
\]

Compute:
(a) A
(b) E(X)
(c) \text{var}(X)
2. A system consists of two lightbulbs connected in series. If the system fails we assume one of the two bulbs failed. The probability that bulb A will work at time t is $P_A(t) = \exp(-t) U(t)$. Similarly, for bulb B: $P_B(t) = \exp(-2t) U(t)$. If the system fails in 4 time-units, what is the probability that bulb A caused the failure?
3. The random variable $X$ is distributed Poisson with parameter $a$. Find the conditional density function if we know that $0 \leq X \leq 2$. 
4. \( X \) is uniformly distributed between zero and unity. We perform the transformation:

\[
Y = \ln(X)
\]
5. \( X \) is distributed as a Beta random variable.

(a) Compute \( E(X^2) \)

(b) Simplify your solution to a ratio of products of factorials when \( b \) and \( c \) are integers.
6. It is easy to generate a uniformly distributed random variable on a computer. Many engineering problems require normal (or Gaussian) random variables. Compute a non-linearity, \( g(x) \), such that

\[
Y = g(X)
\]

is a normal random variable with mean \( \mu \) and variance \( \sigma^2 \) when \( X \) is uniformly distributed between zero and unity.
Some useful relationships:

Beta density: \( f_x(x) = \frac{1}{B(b+1, c+1)} x^b (1-x)^c \) on \( 0 \leq x \leq 1 \)

Normal density: \( f_x(x) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left( \frac{(x-\mu)^2}{2\sigma^2} \right) \) on \(-\infty < x < \infty\)

Poisson density: \( f_x(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta(x-k) \)

Gaussian:
\[
\begin{align*}
F_x(x) &= \frac{1}{2} + \text{erf}\left(\frac{x-\mu}{\sigma}\right) \\
\text{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \, du \\
\Theta(b, c) &= \Gamma(b) \Gamma(c) / \Gamma(b+c) \quad ; \quad \Gamma(n+1) = n! 
\end{align*}
\]

1. A random variable has a probability density function:

\[
f_x(x) = \begin{cases} 
A / x^2 & ; \quad 1 \leq x \leq 2 \\
0 & ; \quad \text{otherwise}
\end{cases}
\]

Compute:

(a) \( A \)

(b) \( E(X) \)

(c) \( \text{var}(X) \)

\[\int_0^2 f_x(x) \, dx = 1 = A \int_0^2 x^{-2} \, dx = A (-x^{-1}) \bigg|_0^2 = A (-\frac{1}{2} + 1) = A \frac{1}{2} \implies A = 2\]

\[(b) \ E(X) = 2 \int_1^2 x \left( \frac{1}{x^2} \right) \, dx = 2 \int_1^2 \frac{dx}{x} = 2 \ln x \bigg|_1^2 = 2 \ln 2\]

\[(c) \ E(X^2) = 2 \int_1^2 x^2 \left( \frac{1}{x^2} \right) \, dx = 2 \]

\[\text{var}(X) = E(X^2) - E^2(X) = 4 - 4 \ln^2 2\]

\[= 4 (1 - \ln^2 2) \approx 2.078\]
2. A system consists of two lightbulbs connected in series. If the system fails, we assume one of the two bulbs failed. The probability that bulb A will work at time \( t \) is \( P_A(t) = \exp(-t) U(t) \). Similarly, for bulb B: \( P_B(t) = \exp(-2t) U(t) \).

If the system fails in 4 time-units, what is the probability that bulb A caused the failure?

\[
\text{CAN USE BAYE'S THEOREM:} \\
P(A/F) = \frac{P(F/A) P(A)}{P(F/A) P(A) + P(F/B) P(B)} \\
A = A \text{ FAILED}, \quad B = B \text{ FAILED} \\
F = \text{SYSTEM FAILED} \\
\text{OBSVIOUSLY:} \quad P[F/A] = P[F/B] = 1 \\
P(A) = 1 - P_A(4) = 1 - e^{-4} \\
P(B) = 1 - P_B(4) = 1 - e^{-8} \\
P(A/F) = \frac{1 - e^{-4}}{(1 - e^{-4}) + (1 - e^{-8})} = 0.495
3. The random variable $X$ is distributed Poisson with parameter $\lambda$. Find the conditional 
density function if we know that $0 \leq X \leq 2$.

$$X \sim f_X(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \delta(x-k)$$

**NOTE:**

$$f_X(x \mid 0 \leq X \leq 2) = f_X(x \mid X \leq 2) = \left\{ \begin{array}{ll}
\frac{f_X(x)}{F_X(2)} & ; \ x \leq 2 \\
0 & ; \ x > 2
\end{array} \right.$$ 

$$F_X(2) = P_r[X \leq 2] = (1 + \lambda + \frac{\lambda^2}{2}) e^{-\lambda}$$

$$f_X(x \mid 0 \leq X \leq 2) = \frac{e^{\lambda}}{1 + \lambda + \frac{\lambda^2}{2}} \left[ e^{-\lambda} \delta(x) + e^{-\lambda} \lambda \delta(x-1) + \frac{\lambda^2}{2} \delta(x-2) \right]$$

$$= \frac{\delta(x) + \lambda \delta(x-1) + \frac{\lambda^2}{2} \delta(x-2)}{1 + \lambda + \frac{\lambda^2}{2}}$$
Find the density function for the random variable, $Y$. $Y = \ln(X)$.

\[ f_Y(y) = \frac{df_X(x)}{dy} = e^y f_X(e^y) \]

Note that $X$ is monotonically $f$ over $0 \leq x \leq 1$.

\[
\begin{align*}
&f_X(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \\
&f_X(e^y) = \begin{cases} 
1 & 0 \leq y < \infty \\
0 & \text{otherwise}
\end{cases} \\
&0 \leq y < \infty
\end{align*}
\]

Thus:

\[
\begin{align*}
f_Y(y) &= \begin{cases} 
1 & 0 \leq y < \infty \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
1 & y \leq 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\int_{-\infty}^{\infty} f_Y(y) \, dy = 1
\]
5. $X$ is distributed as a Beta random variable.

(a) Compute $E(x^k)$.

(b) Simplify your solution to a ratio of products of factorials when $b, c$ are integers.

Note: $f_{X}(x) = \frac{1}{B(b+c+1)} x^{b-1} (1-x)^{c-1} \quad 0 < x < 1$

Thus: $E[x^k] = \int_0^1 x^k f_x(x) \, dx$

Now: $E[x^k] = \frac{1}{B(b+c+1)} \int_0^1 x^{b+k-1} (1-x)^{c-1} \, dx$

$= \frac{\beta(b+c+k+1)}{\beta(b+c+1)} = \frac{\Gamma(b+c+k+2)}{\Gamma(b+c+1)} \\ \frac{\Gamma(b+c+1) }{b+c+n+1} ! \frac{\Gamma(b+c+2)}{(b+c+n+1)!} \frac{(b+c+n+1)!}{(b+n)!}$
6. It is easy to generate a uniformly distributed random variable on a computer. Many engineering problems require normal (or Gaussian) random variables. Compute a non-linearity, \( g(x) \), such that

\[
y = g(x)
\]

is a normal random variable with mean \( \mu \) and variance \( \sigma^2 \) when \( X \) is uniformly distributed between zero and unity.

Assume \( g \) is strictly increasing. If so:

\[
F_Y(y) = F_X[g^{-1}(y)]
\]

where:

\[
F_X(x) = \begin{cases} 
0 & ; x < 0 \\
\frac{1}{2} & ; 0 \leq x < 1 \\
1 & ; x \geq 1 
\end{cases}
\]

thus:

\[
F_X[g^{-1}(y)] = \frac{1}{2} + \text{erf} \left( \frac{y - \mu}{\sigma} \right)
\]

Must be

\[
x = g^{-1}(y) = \frac{1}{2} + \text{erf} \left( \frac{y - \mu}{\sigma} \right)
\]

Solving for \( y = g(x) \):

\[
\text{erf} \left( \frac{y - \mu}{\sigma} \right) = x - \frac{1}{2}
\]

\[
y = g(x) = \sigma \text{erf}^{-1} \left( x - \frac{1}{2} \right) + \mu
\]
A joint pdf has uniform mass over the shown shaded area.

(b) Compute the marginal pdf, $f_Z(z)$. The area under the pdf should be 1.
2. A sample \( X \) is taken from a normal pdf with mean \( \mu \) and variance \( \sigma^2 \). Form the random process
\[
X(t) = Y \cdot e^{-at} \cdot \cos(2\pi f t + \phi(t))
\]
(a) Compute the mean and autocorrelation of \( X(t) \).
(b) If the process is stationary, let \( z(t) \) denote the filter output.
(c) Compute the mean and autocorrelation of \( z(t) \).
3. Let $X(t)$ denote a zero mean random process with
autocorrelation $R_X(t)$. Let $Y(t) = f(t)X(t)$, where
$f(t)$ is a given deterministic function.
(a) Compute the mean of $Y(t)$
(b) Compute the autocorrelation $R_Y(t_1, t_2)$
(c) Is $Y(t)$ stationary in the wide sense? Why or why not?
If $X,Y$ have pdf $e^{-x}U(x)$, form the transformation

$$Z = 2X + Y; \quad V = X + Y$$

(a) Compute the joint density function $f_{Z,V}(z,v)$. 

(b) Sketch the $u,v$ plane and clearly specify the region(s) over which $f_{Z,V}(z,v)$ is not identically zero.
Let \( \bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n \) where the \( X_n \)'s are independent Cauchy random variables with pdf \( f_X(x) = \frac{\alpha}{\pi (x^2 + \alpha^2)} \).

(a) Compute the density of \( \bar{X} \). Recall \( \Phi_X(w) = e^{-\alpha |w|} \).

(b) Comment on the applicability of the central limit theorem to this problem.
In Table 10-2 we show a number of autocorrelations and their transforms. We leave the easy proofs as exercises.

**Comment.** The power spectrum $S(\omega)$ of a process $x(t)$ can be expressed directly in terms of its second-order density $j(x_1,x_2;\tau)$. To this end we introduce the Fourier transform of $j(x_1,x_2;\tau)$ with respect to $\tau$:

$$G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} j(x_1,x_2;\tau)e^{-i\omega\tau} d\tau$$

Since [see (9-7)]

$$R(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1,x_2;\tau) dx_1 dx_2$$

we have

$$|a|^2 R(\tau) = \int_{-\infty}^{\infty} |a|^2 f(x_1,x_2;\tau) dx_1 dx_2$$

and

$$|a|^2 S(\omega) = \int_{-\infty}^{\infty} |a|^2 S(x_1,x_2;\omega) dx_1 dx_2$$

In this case the basic relationships (10-14) and (10-15) take the form

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) \cos \omega \tau \, d\tau \quad R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \cos \omega \tau \, d\omega \quad (10-17)$$

The cross-power spectrum $S_{xy}(\omega)$ of two processes $x(t)$ and $y(t)$ is the Fourier transform of their cross-correlation:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-i\omega \tau} \, d\tau = S_{yx}(\omega) \quad (10-18)$$

The Fourier inversion formula gives

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega)e^{i\omega \tau} \, d\omega$$

and with $\tau = 0$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) \, d\omega = R_{xy}(0) = E[x(t)y^*(t)] \quad (10-19)$$

If $x(t)$ is the voltage across a two-terminal device and $y(t)$ is the resulting input current, then the above equals the expected value of the power delivered to this device.

If the processes $x(t)$ and $y(t)$ are orthogonal (see page 298), then

$$R_{xy}(\tau) = 0 \quad S_{xy}(\omega) = 0$$

In this case [see (10-7)]

$$R_{x+y}(\tau) = R_x(\tau) + R_y(\tau) \quad S_{x+y}(\omega) = S_x(\omega) + S_y(\omega)$$

Table 10-1 shows the correspondence between a process $x(t)$, its autocorrelation $R(\tau)$, and power spectrum $S(\omega)$. The justification follows easily from definition (10-14) and the elementary properties of Fourier transforms [see also (9-85)].
1. A joint pdf has uniform mass over the shown shaded area:

(a) What is the uniform height of $f_{X,Y}(x,y)$?
(b) Compute the marginal pdf, $f_X(y)$.

**Solution**

(a) AREA = $2 \int_0^\infty e^{-x} \, dx = 2$

VOLUME = 1 $\Rightarrow$ HEIGHT = $\frac{1}{2}$

(b) $f_X(y) = \int_{x=0}^{x=\infty} f_{X,Y}(x,y) \, dx$

obviously: $f_X(y) = 0$ for $|y| > 1$

otherwise:

$$f_X(y) = \begin{cases} \int_{x=-1}^{x=1} \frac{1}{2} \, dx & ; 0 \leq y \leq 1 \\ \int_{x=0}^{x=\infty} \frac{1}{2} \ln y \, dx & ; -1 \leq y < 0 \end{cases}$$

or:

$$f_X(y) = \begin{cases} -\frac{1}{2} \ln y & ; 0 \leq y \leq 1 \\ -\frac{1}{2} \ln (-y) & ; -1 \leq y < 0 \\ 0 & ; \text{otherwise} \end{cases}$$

$$= \begin{cases} -\frac{1}{2} \ln |y| & ; |y| \leq 1 \\ 0 & ; |y| > 1 \end{cases}$$
2. A sample $Y$ is taken from a normal pdf with mean $\mu_Y$ and variance $\sigma^2$. Form the random process:

$$X(t) = Y; \quad -\infty < t < \infty$$

(a) Compute the mean and autocorrelation of $X(t)$.

(b) $X(t)$ is the input to a linear time-invariant filter with impulse response $h(t) = e^{-\alpha t}$.

Let $Z(t)$ denote the filter output. Compute the mean and autocorrelation of $Z(t)$.

Solution:

(a) $\mu_X = E[X(t)] = E[Y] = \mu_Y$

$$R_x(t) = E[X(t)X(t+\tau)] = E[Y^2]$$

Since $\sigma^2 = E[Y^2] - \mu_Y^2$, $R_x(\tau) = \sigma^2 + \mu_Y^2$

Yes, the process is stationary.

(b) $S_Z(\omega) = |H(j\omega)|^2 S_x(\omega)$

$$S_x(\omega) = 2\pi(\sigma^2 + \mu_Y^2) \delta(\omega)$$

$$|H(j\omega)|^2 = \frac{2\alpha}{\alpha^2 + \omega^2} \Rightarrow S_Z(\omega) = \left(\frac{\sigma^2}{\alpha}\right)^2 \frac{2\pi(\sigma^2 + \mu_Y^2)}{\alpha^2} \delta(\omega)$$

$$\Rightarrow \Re(\tau) = \frac{4(\sigma^2 + \mu_Y^2)}{\alpha^2}$$

$$\Rightarrow H(0) \Rightarrow \mu_X = \frac{\sigma^2}{\alpha^2} \mu_Y$$
3. Let \( \text{I}(t) \) denote a zero mean random process with

\[
f(t) = E[X(t)] = 0
\]

(a) Compute the mean of \( Y(t) \).

(b) Compute the autocorrelation \( R_y(t, t) \).

(c) Is \( I(t) \) stationary in the wide sense? Why or why not?

\[
R_y(t, t) = E[X(t)X(t)] = f(t)f(t)
\]

Since \( E[X(t)] = 0 \),

\[
R_y(t, t) = E[f(t)f(t)] = 0
\]

Solution:

(a) \( Y(t) = f(t) \).

(b) \( R_y(t, t) = E[f(t)f(t)] = f(t)f(t) \).

(c) No way. \( R_y \) is not strictly a function of \( t \).

The trivial case \( f(t) \) = constant.
q. Both $X$ and $Y$ have pdf's $e^{-x}U(x)$. Form the transformation:

$$\begin{align*}
Z &= 2X + Y \\
V &= X + Y
\end{align*}$$

(a) Compute the joint density function $f_{Z|X}(u, v)$

(b) Sketch the $u-v$ plane and clearly specify the region(s) over which $f_{Z|X}(u, v)$ is not identically zero.

\[\text{Solution}\]

\begin{align*}
&\quad u = 2x + y \\
&\quad v = x + y \\
&\quad x = u - v \\
&\quad y = 2v - u
\end{align*}

\[
|J| = \begin{vmatrix}
2 & 1 \\
1 & 1
\end{vmatrix} = 1
\]

\[
f_{Z|X}(u, v) = |J|e^{-(u-v+2v-u)}U(u-v)U(2v-u)
\]

\[
= e^{-v}U(u-v)U(2v-u)
\]

(b) $U(u-v) = 1 \quad ; \quad u-v > 0$

$U(u) = 1 \quad ; \quad u > v$

$U(2v-u) = 1 \quad ; \quad 2v > u$

$U(v) = 1 \quad ; \quad v > u$
5. Let $\bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$ where the $X_n$'s are independent Cauchy random variables with pdf $f_{X_n}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}$.

(a) Compute the density of $\bar{X}$. Recall $\Phi_X(\omega) = e^{-\alpha |\omega|}$.

(b) Comment on the applicability of the central limit theorem to this problem.

\textbf{Solution:}

\[ \Phi_X(\omega) = E[e^{i\omega \bar{X}}] = E\left[ e^{i\omega \frac{1}{N} \sum_{n=1}^{N} X_n} \right] \]

\[ = E\left[ \prod_{n=1}^{N} e^{i\frac{\omega}{N} X_n} \right] \]

\[ = \prod_{n=1}^{N} E\left[ e^{i\frac{\omega}{N} X_n} \right] \quad \text{since } X_n \text{'s independent} \]

\[ = \left[ E\left( e^{i\frac{\omega}{N} X} \right) \right]^N \quad \text{since identically distributed} \]

\[ = \Phi_X(\omega) = \left[ e^{-\alpha |\frac{\omega}{N}|} \right]^N \]

\[ = e^{-\alpha |\omega|} \Rightarrow f_{\bar{X}}(x) = \frac{\alpha/\pi}{\alpha^2 + x^2} \]

(b) The distribution of $\bar{X}$ is Cauchy, irrespective of the value of $N$. The central limit theorem is thus not applicable here. Note the Cauchy distribution has an infinite second moment so an assumption of proof of the central limit theorem is not fulfilled.
1. We roll three dice: one red, one blue and one yellow. Consider the following events:

A = the red die shows 6
B = the sum of the red and yellow dice is three
C = the sum on all three dice is seven
D = the blue die turns up odd
E = the red die turns up odd
F = the sum of the red and yellow dice is eight

Classify each of the following groups of events as:

Independent (I)
Mutually Exclusive (N)
Neither (N)

No penalty for guessing.

A & B
A & D
A & E
A & F
B & C
B & D
B & E
C & F
D & E
B, [ ] F
John has 6 red and 2 blue socks.
Frank has 2 red and 6 blue socks.
We choose a man at random and take a sock from him at random.
The sock is red.
What is the probability that the man we chose was John?
Let $r$ be an integer. For the gamma distribution, if $b + 1 = r/2$ and $c = 1/2$, the resulting random variable is called $X^2_r$ (chi-squared with $r$ degrees of freedom). Let $X$ come from a distribution with $Y = X^{1/2}$. Compute $P(X > 41)$.
Below is pictured a pdf.
Sketch the conditional density, $f_X(x | 0 \leq x \leq a)$.

\begin{center}
\includegraphics[width=\textwidth]{pdf.png}
\end{center}

Sketch your result here.
The Weibull distribution is defined by:

$$F_X(x) = \left[1 - \exp\left(\frac{x}{\hat{A}}\right)^{\hat{B}}\right] \hat{A}(x)$$

where $\hat{A}$ is the "scale" and $\hat{B}$ the "shape" parameter. Perform the random variable transformation:

$$Y = X^N$$

$Y$ turns out also to be a Weibull random variable with, say, parameters $\hat{\hat{A}}$ and $\hat{\hat{B}}$. Compute these parameters in terms of $A$, $B$ and $N$. 
1. We roll three dice: one red, one blue and one yellow. Consider the following events:

A = the red die shows 6
B = the blue die shows 3
C = the sum of the red and yellow dice is eight
D = the blue die turns up odd
E = the sum of the red and yellow dice is seven

Classify each of the following groups of events as:

Independent (I)
Mutually Exclusive (M)
Neither (N)

\[ \Pr \{E \cap F\} = \Pr \{E\} \cdot \Pr \{F\} \]

- I: A and B
- M: A and C
- N: A and E
- I: B and C
- M: B and D
- N: B and E
- I: C and E
- M: C and D
- N: C and E
- I: D and E
- M: D and F
- N: D and E
- I: E and F
- M: E and G
- N: E and F
- I: F and G
- M: F and H
- N: F and G
- I: G and H
- M: G and I
- N: G and H
- I: H and I
- M: H and J
- N: H and I
- I: J and K
- M: J and L
- N: J and K
- I: K and L
- M: K and M
- N: K and L
- I: L and M
- M: L and N
- N: L and M
- I: M and N
- M: M and O
- N: M and N
- I: N and O
- M: N and P
- N: N and O
- I: O and P
- M: O and Q
- N: O and P
- I: P and Q
- M: P and R
- N: P and Q
- I: Q and R
- M: Q and S
- N: Q and R
- I: R and S
- M: R and T
- N: R and S
- I: S and T
- M: S and U
- N: S and T
- I: T and U
- M: T and V
- N: T and U
- I: U and V
- M: U and W
- N: U and V
- I: V and W
- M: V and X
- N: V and W
- I: W and X
- M: W and Y
- N: W and X
- I: X and Y
- M: X and Z
- N: X and Y
- I: Y and Z
- M: Y and A
- N: Y and Z
- I: Z and A

No penalty for grasping:
For the pdf:

\[ f_x(x) = \begin{cases} \alpha x^2 & ; 0 \leq x \leq 1 \\ 0 & ; \text{otherwise} \end{cases} \]

find:
(a) the constant \( \alpha \)
(b) \( \bar{x} \)
(c) \( \text{var}(X) \)

\( (a) \int_0^1 \alpha x^3 \, dx = 1 = \frac{\alpha}{4} x^4 \big|_0^1 = \frac{\alpha}{4} \Rightarrow \alpha = 4 \)

\( (b) \bar{x} = 4 \int_0^1 x^4 \, dx = \frac{4}{5} x^5 \big|_0^1 = \frac{4}{5} = \frac{4}{5} \)

\( (c) \bar{x}^2 = 4 \int_0^1 x^5 \, dx = \frac{4}{6} x^6 \big|_0^1 = \frac{2}{3} = 0.67 \)

\[ \text{var}(X) = \bar{x}^2 - \bar{x}^2 \]

\[ = \frac{2}{3} - \left( \frac{4}{5} \right)^2 \]

\[ = \frac{2}{3} - \frac{16}{25} = \frac{50 - 3}{1200} \]

\[ = \frac{47}{1200} = 0.039167 \]
John has a red and 2 blue socks.
Frank has 2 red and 6 blue socks.
We choose a man at random and take a sock from him at random.
The sock is red.
What is the probability that the man we chose was John?

\[ J = \text{John}, \quad F = \text{Frank} \]
\[ r = \text{red sock}, \quad b = \text{blue sock} \]

Find \( \Pr[J/r] \)

From Bayes:

\[
\Pr[J/r] = \frac{\Pr[J] \Pr[r/J]}{\Pr(J) \Pr[r/J] + \Pr[F] \Pr[r/F]}
\]

\[
= \frac{\frac{1}{2} \cdot \frac{6}{8}}{\frac{1}{2} \cdot \frac{6}{8} + \frac{1}{2} \cdot \frac{2}{8}}
\]

\[
= \frac{\frac{6}{8}}{\frac{3}{4}} = \frac{3}{4} = 0.75
\]
Let \( r \) be an integer. For the gamma distribution, if \( b = r/2 \) and \( c = 1/2 \), the resulting random variable is called \( \chi^2_r \) (chi-squared with \( r \) degrees of freedom). Let \( X \) come from a distribution with \( r \leq 4 \). Compute \( \Pr(0 \leq X \leq 1) \).

In general:

\[
f_x(x) = \frac{b^x}{\Gamma(b)} x^b e^{-cx} \mu(x)
\]

\[
P = \Pr[0 \leq X \leq 1] = \int_0^1 f_x(x) \, dx
\]

\[
c = \frac{1}{2}, \quad b + 1 = \frac{r}{2} = 2 \Rightarrow b = 1
\]

\[
f_x(x) = \frac{(\frac{1}{2})^2}{\Gamma(2)} x e^{-x/2} \mu(x) = \frac{1}{4} x e^{-x/2} \mu(x)
\]

\[
P = \frac{1}{4} \int_0^1 x e^{-x/2} \mu(x) \, dx
\]

Integration by parts:

\[
u = x, \quad du = \frac{1}{2} e^{-x/2} \, dx
\]

\[
v = -\frac{1}{2} e^{-x/2}, \quad dv = dx
\]

\[
\Rightarrow p = -\frac{1}{2} x e^{-x/2}\bigg|_0^1 + \frac{1}{2} \int_0^1 e^{-x/2} \, dx
\]

\[
= -\frac{1}{2} e^{-1/2} + e^{-1/2} - \left. e^{-x/2}\right|_0^1
\]

\[
= -\frac{1}{2} e^{-1/2} + 1 - e^{-1/2}
\]

\[
= 1 - \frac{3}{2} e^{-1/2} = 0.09020
\]
f(x) looks like this → stretched to yield an area of one. In terms of A, the area equals \( \frac{3}{2} A \Rightarrow A = \frac{3}{2} \). Result sketched above.
The Weibull distribution is defined by:

\[ F_X(x) = \left[ 1 - \exp\left(\frac{x}{A}\right)^B \right] A(x) \]

where \( A \) is the "scale" and \( B \) the "shape" parameter. Perform the random variable transformation:

\[ y = x^A \]

\( y \) turns out also to be a Weibull random variable with, say, parameters \( \hat{A} \) and \( \hat{B} \). Compute these parameters in terms of \( A \), \( B \) and \( N \).

\[
F_y(y) = Pr \left[ Y \leq y \right] = Pr \left[ X^N \leq y \right] \\
\text{over } 0 < x < \infty, \quad y = x^N \text{ is monotonic} \\
\Rightarrow x = y^{1/N}
\]

Thus:

\[
F_y(y) = Pr \left[ X \leq y^{1/N} \right] = F_X(y^{1/N}) \\
= \left[ 1 - e^{-\left(\frac{y^{1/N}}{A}\right)^B} \right] \mu(y^{1/N}) \\
= \left[ 1 - e^{-\left(\frac{y}{A^N}\right)^B} \right] \mu(y) \\
= \left[ 1 - e^{-\left(\frac{y}{A}\right)^B} \right] \mu(y)
\]

Thus:

\[ A = A^N \]
\[ B = B/N \]
1. Let $X_n$, $n = 1, 2, \ldots, N$ denote independent Cauchy random variables distributed thusly:

$$f_{X_n}(x_n) = \frac{\alpha}{\pi(x_n^2 + \alpha^2)} ; n = 1, 2, \ldots, N$$

Form the average: $\bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$. Compute $f_{\bar{X}}(x)$. 

**Hints**

$\int ye^{-ay^2}dy = \frac{-1}{2a}e^{-ay^2}$

$\int_{-\infty}^{\infty} e^{-a|w|}e^{iwx}dw = \frac{2\alpha}{\alpha^2 + x^2}$

$\int xe^{-ax^2}dx = -e^{-ax^2} \left[ x + \frac{1}{a} \right]$
Let $\{X_n\}_{n=1}^\infty$ be independent identically distributed random variables with mean zero and unit variance. We form the average $\overline{X} = \frac{1}{N} \sum_{n=1}^N X_n$.

Find a lower bound on the probability that $\overline{X}$ lies between $-a$ and $a$, for $a > 0$. 

$\text{Var}(\overline{X}) = \frac{\text{Var}(X)}{N} = \frac{1}{N}$.
I and $Y$ are independent zero-mean normal random variables with variance $\sigma^2$. Let

\[
Z = \frac{X}{Y}
\]

compute $f_Z(z)$. Let
Let $X$ and $Y$ denote positive random variables, i.e.
\[ f_{XY}(x,y) = f_{XY}(x,y) \mu(x) \mu(y) \]
where $\mu(\cdot)$ denotes the unit step. Let
\[ U = X^Y, \quad V = X^{-2} \]
Find $f_{UV}(u,v)$ in terms of $f_{XY}(x,y)$.
Include appropriate limits
Consider three statistics: \( X_0, X_1, \) and \( X_2 \). Given:

\[ X_1 \text{ and } X_2 \text{ are orthogonal,} \]

\[ E[X_0X_1] = E[X_0X_2] = 0, \]

\[ E[X_1^2] = 2, \quad E[X_2^2] = 3 \]

find the best linear estimate of \( X_0 \) in terms of \( X_1 \) and \( X_2 \).
Consider the conditional density:

\[ f_{Y|X}(y|x) = \frac{1}{x} h(x) e^{-xy} \mu(y) \]

where \( \mu(y) \) denotes the unit step.

(a) Compute the function \( h(x) \).

(b) Statistics \( X \) and \( Y \) are taken from \( f_{XY}(x,y) \), the joint density from which \( f_{Y|X}(y|x) \) above was obtained. The value of \( X \) was 2. What is our corresponding m.s.e. of \( Y \)? Give a number.
1. Let $X_n, n = 1, 2, \ldots, N$ denote independent Cauchy random variables distributed thusly:

$$f_{X_n}(x_n) = \frac{\alpha}{\pi (\alpha^2 + x_n^2)} ; n = 1, 2, \ldots, N$$

Form the average: $\bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$. Compute $f_{\bar{X}}(x)$.

From hint:

$$\Phi_{X_n}(\omega) = e^{-\alpha |\omega|}$$

Thus:

$$\Phi_{\bar{X}}(\omega) = E[e^{i\omega \bar{X}}] = \Phi_{X_n}^N(\frac{\omega}{N})$$

showed in class for average.

$$= \left[ e^{-\alpha \frac{|\omega|}{N}} \right]^N$$

$$= e^{-\alpha |\omega|}$$

same thing!

Hence: $\bar{X}$ is Cauchy:

$$f_{\bar{X}}(x) = \frac{\alpha}{\pi (\alpha^2 + x^2)}$$

Note: This was prob. 8-19 on p. 275 of text.
N independent identically distributed random variables \( \{X_n \mid n = 1, 2, 3, \ldots, N \} \) are zero mean and unit variance. We form the average:

\[
\overline{X} = \frac{1}{N} \sum_{n=1}^{N} X_n
\]

Find a lower bound on the probability that \( \overline{X} \) lies between \( -a \) and \( a > 0 \).

\[\overline{X} \text{ has mean } 0 \quad \text{and variance } \frac{1}{N} \sigma_x^2.\]

From Chebyshev:

\[
Pr[|\overline{X}| < k\sigma] \geq 1 - \frac{1}{k^2}
\]

\[k\sigma = a, \quad \overline{X} = \overline{X}
\]

\[
Pr[|\overline{X}| < a] \geq 1 - \left(\frac{\sigma_x}{a}\right)^2 = 1 - \left(\frac{1}{N a^2}\right)^2.
\]

Note: \( Pr[|\overline{X}| < a] \xrightarrow{n \to \infty} 1 \)
I and Y are independent zero-mean normal random variables with variance $\sigma^2$. Let $Z = \frac{X}{Y}$. Compute $f_Z(z)$. From class work:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \frac{1}{y} dy$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Thus:

$$f_Z(z) = \frac{1}{2\pi \sigma^2} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} \frac{1}{y} dy$$

$$= \frac{1}{\pi \sigma^2} \int_{-\infty}^{\infty} \frac{1}{y} \left( e^{-\frac{x^2+y^2}{2\sigma^2}} \right) dy$$

$$= \frac{1}{\pi \sigma^2} \int_{-\infty}^{\infty} \left( e^{-\frac{y^2}{2\sigma^2}} \right) dy$$

$$= \frac{1}{\pi \sigma^2 \sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi \sigma^2}}$$

$$\Rightarrow f_Z(z) \sim \text{Cauchy}$$
Let $X$ and $Y$ denote positive random variables, i.e.,

$$f_{X,Y}(x,y) = f_{Z,Z}(x,y,\mu(x),\mu(y)).$$

Find $f_{X,Y}(u,v)$, in terms of $f_{Z,Z}(x,y,\mu(x),\mu(y))$.

Include appropriate limits.

From transformation, it is obvious that both

$W(x,y) =\begin{cases} 0 & \text{if } U \leq 1 \\ 2 & \text{if } U > 1 \end{cases}$

$W(x,y) =\begin{cases} 0 & \text{if } U \leq 1 \\ 2 & \text{if } U > 1 \end{cases}$

For $U > 1$ and $W(x,y) > 0$, the shaded region:

For $U > 1$ and $W(x,y) > 0$, the shaded region:
Consider three statistics: $X_0$, $X_1$, and $X_2$. Given:

- $X_1$ and $X_2$ are orthogonal,
- $E[X_0 X_1] = E[X_0 X_2] = 6$, $E[X_1^2] = 2$, $E[X_2^2] = 3$

find the best linear estimate of $X_0$ in terms of $X_1$ and $X_2$.

\[ X_0 = a_1 X_1 + a_2 X_2 \]

where $a_1 \neq a_2$ satisfy

\[
\begin{align*}
R_{01} &= a_1 R_{11} + a_2 R_{12} \\
R_{02} &= a_1 R_{12} + a_2 R_{22}
\end{align*}
\]

or:

\[
\begin{bmatrix}
6 \\
6
\end{bmatrix} =
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
\]

But:

\[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}^{-1} =
\begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{3}
\end{bmatrix}
\]

\[
\Rightarrow
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
6 \\
6
\end{bmatrix} =
\begin{bmatrix}
3 \\
2
\end{bmatrix}
\]

linear

hence, m.s.e. is

\[
\hat{X}_0 = 3X_1 + 2X_2
\]
Consider the conditional density:
\[ f_{Y|X}(y|x) = \frac{1}{X} h(x) e^{-xy} \mu(y) \]
where \( \mu(y) \) denotes the unit step.

(a) Compute the function \( h(x) \).

(b) Statistics \( X \) and \( Y \) are taken from \( f_{XY}(x,y) \): the joint density from which \( f_{Y|X}(y|x) \) above was obtained. The value of \( X \) was 2. What is our corresponding m.s.e. of \( \hat{Y} \)? Give a number.

\[ (a) \text{ for all } x: \int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1 = \frac{1}{x} h(x) \int_{0}^{\infty} e^{-xy}dx = \frac{1}{x^2} h(x) \implies h(x) = x^2 \]

(b) m.s.e. estimate is:
\[ \hat{Y} = E(Y|X) = E(Y|X) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy \]
\[ = x \int_{0}^{\infty} ye^{-xy}dy \]
\[ = x \left[ ye^{-xy} \right]_{y=0}^{\infty} + \int_{0}^{\infty} e^{-xy}dy = \frac{1}{y} \]
for \( X = 2 \) \( \implies \hat{Y} = \frac{1}{2} \)
1. Given the characteristic function:

\[ \Phi_X(\omega) = \frac{A}{\cos(a\omega)} \]

where "a" is a parameter, compute:

(a) A  (b) \(E[X]\)  (c) \(\text{var}(X)\)
The "lognormal density" is so named because it is the density of \( X \) if \( \ln X \) is normally distributed with, say, mean \( \mu \) and variance \( \sigma^2 \). Compute the density function of a lognormal random variable. Remember limits.
The spectral density of zero mean white noise is:
\[ S_X(\omega) = \text{const}; \quad \forall \omega \]

Let \( X(t) \) denote such a process with \( \text{const.} = 1 \).

We pass \( X(t) \) thru a filter with transfer function:
\[ H(\omega) = \begin{cases} \gamma \cos \omega ; & |\omega| \leq \pi \\ 0 ; & \text{otherwise} \end{cases} \]

Compute the output signal noise level
Let $A$ and $\Theta$ denote independent random variables. $A$ has mean "a" and variance $\sigma_A^2$. $\Theta$ has characteristic function $\Phi_\Theta(\omega)$. Form the random amplitude-phase rotating phasor:

$$X(t) = AE^{j[2\pi ft + \Theta]}; \quad f = \text{given constant}$$

(a) Compute the mean and autocorrelation of $X(t)$ in terms of $a$, $\sigma_A^2$ and $\Phi_\Theta(\omega)$

(b) Is the process stationary?
Consider the joint density:

\[ f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{shaded area} \\ 0 & \text{otherwise} \end{cases} \]

The corresponding radial distance is:

\[ R = \sqrt{X^2 + Y^2} \]

Compute:

\[ E[R^2] = \int_0^{\sqrt{2}} r^2 f_R(r) \, dr \]
Let \( \{X_n\}_{n=1}^{N} \) denote identically distributed independent random variables with unknown mean \( \mu \) and known variance \( \sigma^2 \). We form the average:

\[
\bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n
\]

Assume \( N \) is sufficiently large for application of the central limit theorem.

(a) What is the probability that \( \bar{X} \) lies within \( \alpha \) standard deviations of \( \mu \)?

(b) How many samples do we need to be assured that there is about a 99\% chance \( \bar{X} \) is within \( \frac{1}{100} \) of a standard deviation from \( \mu \)?
Given the stochastic differential equation:
\[ \sum_{n=0}^{N} a_n Y_n^{(n)}(t) = X(t) ; Y_n^{(n)}(0) = 0 \text{, } n = 0, 1, \ldots, N-1 \]
we compute the cross correlation by solving the deterministic differential equation:
\[ \sum_{n=0}^{N} a_n \left( \frac{d}{dt} \right)^n R_{XY}(t_1, t_2) = R_X(t_1, t_2) \]
with initial conditions: \( \left( \frac{d}{dt} \right)^n R_{XY}(t_1, 0) = 0 \text{ for } n = 0, 1, \ldots, N-1 \). DERIVE the differential equation we must solve to find \( R_X(t_1, t_2) \). Don't forget the initial conditions.
Let $X(t)$ denote a stochastic process with mean $\mu_X$ and autocorrelation $R_X(\tau)$. $X(t)$ is ergodic in the mean. Let $A$ denote a random variable with mean $\mu_A$ and variance $\sigma_A^2$. $A$ and $X(t)$ are uncorrelated. Define:

$$Y(t) = A + X(t)$$

Is $Y(t)$ ergodic in the mean? Show your work.
Solutions to Final Examination:

1. Given the characteristic function:
   \[ \Phi_Y(\omega) = \frac{A}{\cosh(\omega)} \]
   where \( a \) is a parameter, compute:
   
   (a) \( A \) 
   (b) var(\( X \)) 
   (c) \( \text{var}(\hat{\theta}) \)

2. \( \Phi_X(0) = \frac{d}{\omega} \Phi_X(0) = \frac{d}{\omega} \cosh^2(\omega) \phi_0(\omega) \)
   \( o = \Phi_X(0) \)

3. \( \hat{\theta} = \int \frac{\theta}{\hat{\theta}} \hat{\theta} \)
   \( \text{var}(\hat{\theta}) = \frac{\theta}{\hat{\theta}} \hat{\theta} \)

4. \( \text{var}(\hat{\theta}) = \frac{\theta}{\hat{\theta}} \hat{\theta} \)
The "lognormal density" is so named because it is the density of $X$ if $\ln X$ is normally distributed with, say, mean $\mu$ and variance $\sigma^2$. Compute the density function of a lognormal random variable. Remember limits.

\[ Y = \ln X \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y - \mu)^2}{2\sigma^2}} \]

\[ \frac{dy}{dx} = \frac{1}{x} \]

\[ \Rightarrow f_x(x) \sim \frac{1}{x \sqrt{2\pi}\sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \frac{1}{\mu(x)} \]
The spectral density of zero mean white noise is:

\[ S_X(w) = \text{const} \text{; all } w \]

Let \( X(t) \) denote such a process with \( \text{const} = 1 \). We pass \( X(t) \) through a filter with transfer function:

\[ H(w) = \begin{cases} \cos(w) & \text{if } |w| \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases} \]

Compute the output signal noise level:

\[ S_Y(w) = S_X(w)|H(w)|^2 \]

\[ \begin{align*}
S_Y(w) &= \int_0^\pi \cos^2(w) S_X(w) w^{-\frac{1}{2}} \, dw \\
&= \frac{1}{4\pi} \int_0^\pi \left[ 1 + \cos(2w) \right] \frac{1}{w^{\frac{1}{2}}} \, dw \\
&= \frac{1}{4\pi} \left[ \int_0^\pi \frac{1}{w^{\frac{1}{2}}} \, dw - \pi \right] \\
&= \frac{1}{4\pi} \left[ 2\pi^\frac{1}{2} - \pi \right] \\
&= \frac{\pi^{\frac{1}{2}}}{4\pi} \\
&= \frac{1}{4\pi^{\frac{1}{2}}} \\
&= \frac{1}{4} 
\end{align*} \]
Let \( A \) and \( \Theta \) denote independent random variables, \( A \) has mean \( \mu \) and variance \( \sigma_A^2 \). \( \Theta \) has characteristic function \( \Phi_\Theta(\omega) \). Form the random amplitude-phase rotating phasor:

\[
X(t) = Ae^{j(2\pi ft + \Theta)} ; \quad f = \text{given constant}
\]

(a) Compute the mean and autocorrelation of \( X(t) \) in terms of \( \mu, \sigma_A^2 \) and \( \Phi_\Theta(\omega) \).

(b) Is the process stationary?

\[
(a) \quad M_x(t) = E[X(t)] = E[Ae^{j(2\pi ft + \Theta)}] = E[A] E[e^{j(2\pi ft + \Theta)}] \quad \text{follows from independence}
\]

Since \( \Phi_\Theta(\omega) = E[e^{j\omega \Theta}] \),

\[
M_x(t) = a \Phi_\Theta(0) e^{j2\pi ft}
\]

\[
R_x(t_1, t_2) = E[A^2 e^{j[2\pi f(t_1 - t_2) + 2\Theta]}] = [\sigma_A^2 + a^2] e^{j2\pi f(t_1 - t_2)} E[e^{j2\Theta}] \Rightarrow R(t_1 - t_2) = R(t) = (\sigma_A^2 + a^2) \Phi_\Theta(2) e^{j2\pi ft}
\]

(b) Mean changes with \( t \)

\( \Rightarrow \) not stationary
Consider the joint density:

\[ f(x, y) = \begin{cases} \frac{1}{2} & 0 < x < 1, \\
0 & \text{otherwise} \end{cases} \]

The corresponding radial distance is:

\[ R = \sqrt{x^2 + y^2} \]

Compute:

\[ E[R^2] = \int_0^1 \int_0^{\sqrt{1-x^2}} f(r)(r)(r^2) \, dr \, dy \]

\[ = \frac{1}{2} \int_0^1 \left[ \int_0^{\sqrt{1-y^2}} (x^2+y^2) \, dx \right] dy \]

\[ = \frac{1}{2} \int_0^1 \left[ \frac{1}{2} (x^3+y^2) \right]_0^{\sqrt{1-y^2}} dy \]

\[ = \frac{1}{2} \int_0^1 \left[ \frac{1}{2} \left( 1 - y^2 \right)^{3/2} y^2 \right] \, dy \]

\[ = \left. \frac{1}{2} \left[ \frac{3}{12} y^3 - \frac{1}{16} y^4 \right] \right|_0^1 \]

\[ = \frac{1}{2} \left( \frac{3}{12} - \frac{1}{16} \right) \]

\[ = \frac{1}{12} \left( \frac{1}{2} \right) \]

\[ = \frac{1}{12} \frac{1}{2} \]

\[ = \frac{1}{24} \]
Let \( \{X_n\}_{n=1}^{N} \) denote identically distributed independent random variables with unknown mean \( \mu \) and known variance \( \sigma^2 \). We form the average:

\[
\overline{X} = \frac{1}{N} \sum_{n=1}^{N} X_n
\]

Assume \( N \) is sufficiently large for application of the central limit theorem.

(a) What is the probability that \( \overline{X} \) lies within \( \alpha \) standard deviations of \( \mu \)?

(b) How many samples do we need to be assured that there is about a 99% chance \( \overline{X} \) is within \( \frac{\alpha}{100} \) of a standard deviation from \( \mu \)?

From central limit theorem, \( \overline{X} \) is approximately distributed normal with mean \( \mu \) and variance \( \sigma^2 \).

(a) We want to find:

\[
p = \Pr \left[ \mu - \alpha \sigma \leq \overline{X} \leq \mu + \alpha \sigma \right] = \Pr \left[ -\alpha \sqrt{\frac{N}{N^2}} = z < \alpha \sqrt{\frac{N}{N^2}} \right] = 2 \operatorname{erf} \left( \alpha \sqrt{\frac{N}{N^2}} \right)
\]

(b) \( p = 0.99 \), \( \alpha = \frac{1}{100} \), \( N = ? \)

\[
0.495 = \operatorname{erf} \left( \frac{\sqrt{N}}{100} \right)
\]

From \( \operatorname{erf} \) table:

\[
\frac{\sqrt{N}}{100} \approx 2.60
\]

\[
N \approx 260 \Rightarrow N = (260)^2 
\]

\( \approx 67,600 \) samples
Given the stochastic differential equation:
\[ \sum_{n=0}^{N} a_n Y^{(n)}(t) = X(t) \quad Y^{(n)}(0) = 0, \quad n = 0, 1, \ldots, N-1 \]
we compute the cross correlation by solving the deterministic differential equation:
\[ \sum_{n=0}^{N} a_n \left( \frac{X}{st_2} \right)^n R_{XX}(t_1, t_2) = R_{XY}(t_1, t_2) \]
with initial conditions: \( \left( \frac{X}{st_2} \right)^n R_{XY}(t_1, 0) = 0 \quad \forall \quad n = 0, 1, \ldots, N-1 \). **DERIVE** the differential equation we must solve to find \( R_{XY}(t_1, t_2) \). Don't forget the initial conditions.

From top equation:
\[ \sum_{n=0}^{N} a_n Y^{(n)}(t_1) Y(t_2) = X(t_1) Y(t_2) \quad (*) \]
Since:
\[ R_{XY}(t_1, t_2) = E[Y(t_1) Y(t_2)] \]
It follows that:
\[ \left( \frac{X}{st_1} \right)^n R_{XY}(t_1, t_2) = E[Y^{(n)}(t_1) Y(t_2)] \]
Taking \( E(\cdot) \) of both sides of \((*)\) gives:
\[ \sum_{n=0}^{N} a_n \left( \frac{X}{st_1} \right)^n R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) \]
To get initial conditions, note:
\[ \left( \frac{X}{st_1} \right)^n R_{XY}(0, t_2) = E[Y^{(n)}(0) Y(t_2)] \]
But \( Y^{(n)}(0) = 0 \) and the initial conditions are:
\[ \left( \frac{X}{st_1} \right)^n R_{XY}(0, t_2) = 0 \quad \forall \quad n = 0, 1, \ldots, N-1 \]
Let $X(t)$ denote a stochastic process with mean $\mu_x$ and autocorrelation $R_x(\tau)$. $X(t)$ is ergodic in the mean. Let $A$ denote a random variable with mean $a$ and variance $\sigma_a^2$. $A \perp X(t)$ are uncorrelated. Define:

$$Y(t) = A + X(t)$$

Is $Y(t)$ ergodic in the mean? Show your work.

$$\eta_T = \langle Y \rangle = \frac{1}{2T} \int_{-T}^{T} [X(t) + A] dt$$

$$\Rightarrow E\langle Y \rangle = \frac{1}{2T} \int_{-T}^{T} [\mu_X + a] dt = \mu_X + a$$

This is same as:

$$E[Y(t)] = E[X(t) + A] = \mu_X + a = E\langle Y \rangle$$

Criterion #1 for ergodicity checks.

Second is: Does $\lim_{T \to \infty} \eta_T = 0$ where:

$$\sigma_{\eta_T}^2 = \frac{1}{T} \int_0^{2T} \left( 1 - \frac{x}{2T} \right) \left( R_x(\tau) - \eta_x^2 \right) d\tau$$

Since $\eta_y = \eta_x + a$, and

$$R_x(\tau) = E\left[ (X(t) + A)(X(t+\tau) + A) \right]$$

$$= R_x(\tau) + 2a \mu_X + (\sigma_a^2 + a^2)$$

we have:

$$\sigma_{\eta_T}^2 = \frac{1}{T} \int_0^{2T} \left( 1 - \frac{x}{2T} \right) \left[ R_x(\tau) + 2a \mu_X + (\sigma_a^2 + a^2) \right] d\tau$$

Since $X$ is ergodic we know:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{2T} \left( 1 - \frac{x}{2T} \right) \left( R_x(\tau) - \eta_x^2 \right) d\tau = 0$$

Thus:

$$\lim_{T \to \infty} \frac{\sigma_{\eta_T}^2}{\sigma_a^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^{2T} \left( 1 - \frac{x}{2T} \right) \sigma_a^2 d\tau = \sigma_a^2$$

Thus, $Y$ is not ergodic unless $\sigma_a^2 = 0$
Test Problem:
Consider the joint pdf \( f(x,y) = A y^c e^{-\alpha y} \exp(-bxy) u(x)u(y) \):
(a) Compute \( A \).
(b, c) Find the marginal densities:
\[ f_X(x) \text{ and } f_Z(y) \]
(include limits)
(d) Under what condition(s) are the moments
\[ E[X^n Y^m] = m_{nm} \text{ not finite.} \]
(e) Find \( E[Z^m] = m_{zm} \)
(f) With \( d > 0 \), define
\[ Z = \frac{X + d}{X} \]
Find \( F_Z(z) \) when \( C = 1 \)

I, (print your name)

have neither given assistance nor received help on this exam. The work is mine alone. Any reference source is noted in my work.

\[ \text{sign} \]
\[ \text{date signed} \]
(Hand in this sheet with your work)
Solutions: EE505 Summer '83

(a) \[ 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy \]
\[ = A \int_{y=0}^{\infty} y^c e^{-ay} \int_{x=0}^{\infty} e^{-bxy} \, dx \, dy \]
\[ = A \int_{y=0}^{\infty} y^c e^{-ay} \frac{1}{by} \, dy \]
\[ = A \frac{1}{b} \int_{y=0}^{\infty} y^{c-1} e^{-ay} \, dy \]

Let \( \xi = ay \Rightarrow y = \frac{\xi}{a} \)

\[ 1 = \frac{A}{b} \int_{\xi=0}^{\infty} \left( \frac{\xi}{a} \right)^{c-1} e^{-\xi} \frac{d\xi}{a} \]
\[ = \frac{A}{b a^c} \int_{0}^{\infty} \xi^{c-1} e^{-\xi} d\xi = \frac{A}{b a^c} \Gamma(c) \]
\[ \Rightarrow A = b a^c / \Gamma(c) \]

(b) \( f_{X,Y}(x,y) = \frac{ba^c}{\Gamma(c)} y^c e^{-ay} e^{-bxy} u(x)u(y) \)

\( f_X(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \)
\[ = \frac{ba^c}{\Gamma(c)} y^c e^{-ay} \left[ \int_{x=0}^{\infty} e^{-bxy} \, dx \right] u(y) \]
\[ = \frac{ba^c}{\Gamma(c)} y^c e^{-ay} \left[ \frac{1}{by} \right] u(y) \]
\[ = \frac{a^c}{\Gamma(c)} y^{c-1} e^{-ay} u(y) \]
(c) \( f_X(x) = \frac{ba^c}{\Gamma(c)} \int_0^\infty y^c e^{-ay} e^{-bxy} dy \)

\[ = \frac{ba^c}{\Gamma(c)} \int_0^\infty y^c e^{-(a+bx)y} dy \]

Let \( \varphi = (a+bx)y \):

\[ f_X(x) = \frac{ba^c}{\Gamma(c)} \int_0^\infty \left( \frac{\varphi}{{a+bx}} \right)^c e^{-\varphi} \frac{d\varphi}{a+bx} u(x) \]

\[ = \frac{ba^c \Gamma(c+1)}{\Gamma(c)(a+bx)^{c+1}} u(x) \]

\[ = \frac{ba^c \Gamma(c+1)}{(a+bx)^{c+1}} u(x) \]

(d) \( m_{nm} = \frac{ba^c}{\Gamma(c)} \int_0^\infty x^n \int_0^\infty y^{c+m} e^{-ay} e^{-bxy} dy \)

\[ = \frac{ba^c}{\Gamma(c)} \int_0^\infty x^n \int_0^\infty y^{c+m} e^{-y(a+bx)} dy \]

Let \( \varphi = (a+bx)y \) in \( y \) integral:

\[ m_{nm} = \frac{ba^c}{\Gamma(c)} \int_0^\infty x^n \int_0^\varphi \left( \frac{\varphi}{a+bx} \right)^{c+m} e^{-\varphi} \frac{d\varphi}{a+bx} \]
The integrand is bounded. Divergence occurs only if from the tail. Consider

\[
\lim_{x \to \infty} x^{n-r+1} \int_0^\infty b X^r x^n dx = b \int_0^\infty X^{n-r+1} x^n dx
\]

Thus

\[
\begin{align*}
I &= \int_0^\infty x^n dx \\
&= b \int_0^\infty X^{n-r+1} x^n dx \\
&= \begin{cases} \\
0 & n - r + 1 < 0 \\
1 & n - r + 1 = 0 \\
\frac{\Gamma(r)}{\Gamma(c)} & n - r + 1 > 0 
\end{cases}
\end{align*}
\]

for

\[
\begin{align*}
0 &< n - r + 1 \\
n - c - m - 1 &< 0 \\
n - m &< 0
\end{align*}
\]
(e) From (b):

\[ f_X(y) = \frac{a^c}{\Gamma(c)} y^{c-1} e^{-ay} u(y) \]

\[ E[X^m] = \frac{a^c}{\Gamma(c)} \int_0^\infty y^m e^{-ay} dy \]

\[ \xi = ay \]

\[ E[X^m] = \frac{a^c}{\Gamma(c)} \int_0^\infty \left( \frac{\xi}{a} \right)^m e^{-\xi} \frac{d\xi}{a} \]

\[ = \frac{a^c}{\Gamma(c) a^{m+c}} \int_0^\infty \xi^{m+c-1} e^{-\xi} d\xi \]

\[ = \frac{a^{-m}}{\Gamma(c)} \Gamma(m+c) \quad \text{(a}\ \text{note}\ \text{is}\ \gamma\ \text{pdf}) \]

(f) \[
F_Z(z) = P_r[Z \leq z]
= P_r\left[\frac{Z+d}{X} \leq z\right]
= P_r[Y + d \leq Xz]
= P_r[Y \leq Xz - d]
\]
\[ F_2(z) = \int \int_{\mathbb{R}^2} f_{x,y}(x,y) \, dx \, dy \]

\[ = \frac{b a c}{\Gamma(c)} \int_0^\infty y^{c-1} e^{-a y} \int_0^{y+d} e^{-b x y} \, dx \, dy \]

\[ = \frac{b a c}{\Gamma(c)} \int_0^\infty y^{c-1} e^{-a y} \frac{1}{b y} e^{-b y \left( \frac{y+d}{a} \right)} \, dy \]

\[ = \frac{a c}{\Gamma(c)} \int_0^\infty y^{c-1} e^{-\left( a + \frac{b d}{a} \right) y + b \frac{y^2}{a}} \, dy \]

\[ = \frac{a c}{\Gamma(c)} \int_0^\infty y^{c-1} e^{-\frac{b}{a} \left[ \frac{a y^2}{b} + d + y^2 \right]} \, dy \]

\[ = \frac{a c}{\Gamma(c)} \int_0^\infty y^{c-1} e^{-\frac{b}{a} \left[ y + \frac{1}{2} \left( \frac{a y^2}{b} + d \right) \right]^2} \, dy \]  

\[ e^{\frac{b}{4a} \left( \frac{a y^2}{b} + d \right)^2} \]

For \( c = 1 \):

\[ F_2(z) = \frac{a c}{\Gamma(c)} \int_0^\infty e^{-\frac{b}{a} \left( y + \frac{1}{2} \left( \frac{a y^2}{b} + d \right) \right)^2} \, dy \]  

\[ \frac{1}{2\sigma^2} = \frac{b}{a} \Rightarrow \sigma = \sqrt{\frac{a}{2b}} \]

Let \( y = \frac{y + \frac{1}{2} \left( \frac{a y^2}{b} + d \right)}{\sqrt{\frac{a}{2b}}} \)
\[
\frac{\gamma}{1 + \frac{\beta}{\gamma}} = \gamma - \frac{\beta}{\gamma + \Delta}
\]

\[
\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi}
\]

\[
F_\frac{3}{2}(\varepsilon) = \sqrt{\frac{\pi}{a}} e^{-\frac{a^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{\beta x^2}{\gamma} + \frac{\gamma}{\beta} x^2} \, dx
\]

\[
= \frac{\gamma}{\gamma + \Delta} \int_{-\infty}^{\infty} e^{-\frac{\beta x^2}{\gamma} + \frac{\gamma}{\beta} x^2} \, dx
\]

\[
= 2a \sqrt{\frac{\pi b}{a}} e^{-\frac{a^2}{2b}} \frac{\gamma}{\gamma + \Delta}
\]
1. A random variable $X$ has a characteristic function:

$$
\Phi_X(\omega) = A \cos^2(aw)
$$

where $a$ is a given parameter. Compute:

(a) $A$
(b) $E(X)$
(c) $\text{var}(X)$
Sometimes the expected value of a random variable is not such a good estimate. For example, let $X$ be a Poisson random variable with parameter $\lambda = 1$. Let $Y = (-1)^X$.

Compute $E(Y)$ and comment.
Let $X$ be a gamma random variable with $n$ an integer:

$$f_X(x) = \frac{c^{n+1}}{n!} x^n \exp(-cx) U(x)$$

Let $(X_n)_{n=1,2,...,N}$ be iid samples from this density. Define

$$\bar{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$$

Find the density function for the average, $f_{\bar{X}}(x)$. 

Let $P$ and $Q$ denote independent random variables both uniformly distributed on the interval from zero to unity. Define the process:

$$X(t) = P e^{-Qt} U(t)$$

Find:

(a) $E(X(t))$
(b) $R(t_1, t_2)$
(c) $\text{var} X(t)$
We draw $N$ iid samples from a shifted Laplacian random variable with mean $\mu$ and variance $\sigma^2$. Give an approximation of the density function for the average of these numbers. $N \gg 1$
$X(t)$ is a stationary random process with mean $\mu$ and autocorrelation

$$R(\tau) = \frac{x^2}{\mu^2} \exp(-a|\tau|)$$

where $a$ is a specified parameter. What percentage of the time can we expect $X(t)$ to lie below a given threshold, $T$?
In our take-home problem last week, we found that the joint density,

\[ f_{XY}(x,y) = 8 \, y^2 \, e^{-2y} \, e^{-2xy} \, U(x) \, U(y) \]

had a marginal density

\[ f_y(y) = 4 \, y \, e^{-2y} \, U(y). \]

Given that \( \sum_1^2 y = 2 \), what is a good estimate of \( \bar{X} \)?
1. A random variable $X$ has a characteristic function:

$$
\Phi_X(\omega) = A \cos^2(\alpha \omega)
$$

where $\alpha$ is a given parameter. Compute:

(a) $A$
(b) $E(X)$
(c) $\text{var}(X)$

\[\begin{align*}
(a) & \quad \Phi_X(0) = 1 \implies A = 1 \\
(b) & \quad \Phi_X(\omega) = \frac{1}{2} (1 + \cos 2\alpha \omega) \\
& \quad \frac{d\Phi_X}{d\omega} = -\frac{1}{2} 2\alpha \sin 2\alpha \omega = -\alpha \sin 2\alpha \omega \\
& \quad \frac{d\Phi_X}{d\omega}(0) = jE[X] = 0 \implies E(X) = 0 \\
(c) & \quad \frac{d^2\Phi_X}{d\omega^2} = -\alpha (2\alpha) \cos 2\alpha \omega \\
& \quad \frac{d^2\Phi_X}{d\omega^2}(0) = jE(X^2) = -2\alpha^2 \implies E(X^2) = 2\alpha^2 \\
\text{var} X & = E(X^2) - E^2(X) \\
& = 2\alpha^2
\end{align*}\]
Sometimes the expected value of a random variable is not such a good estimate. For example, let $X$ be a Poisson random variable with parameter $a = 1$. Let $Y = (-1)^X$.

Compute $E(Y)$ and comment.

$$E[Y] = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-a} a^k}{k!}$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{(e^{-1} a)^k}{k!} = e^{-1} e^{-1} = e^{-2} \approx 0.135$$

**COMMENT:** $Y$ is always $\pm 1$. 
Let $X$ be a gamma random variable with $n$ an integer:

$$f_{X}(x) = \frac{c^{n+1}}{n!} x^n \exp(-cx) \ \text{U}(x)$$

Let $(X_n)_{n=1,2,...,N}$ be iid samples from this density. Define

$$\overline{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$$

Find the density function for the average, $f_{\overline{X}}(x)$.

$$\Phi_{\overline{X}}(\omega) = E[e^{j\omega \overline{X}}] = E\left[ e^{j\omega \frac{1}{N} \sum_{n=1}^{N} X_n} \right]$$

$$= E\left[ \prod_{n=1}^{N} e^{j\omega X_n/N} \right]$$

$$= E^n \left[ e^{j\omega X_n/N} \right]$$

$$= \Phi^n_{X}\left( \frac{\omega}{N} \right)$$

From p.154 of text:

$$\Phi_{X}(\omega) = \left( \frac{c}{c-j \omega} \right)^{n+1}$$

$$\Rightarrow \Phi_{\overline{X}}(\omega) = \left( \frac{c^{N(n+1)}}{(c-j \omega)^{N(n+1)}} \right) ; \quad \hat{n} + 1 = \frac{N(N+1)}{N+1} \frac{C^n}{\hat{n}}$$

From Fourier transform scaling theorem:

$$f_{\overline{X}}(x) = N \frac{C^{\hat{n}+1}}{\hat{n}!} (NX)^{\hat{n}} e^{-c(NX)} U(x)$$

$$= N \frac{C^{N(n+1)}}{(N(n+1))!} (NX)^{Nn+N+1} e^{-c(NX)} U(x)$$
Let P and Q denote independent random variables both uniformly distributed on the interval from zero to unity. Define the process:

\[ X(t) = P \cdot e^{-Qt} \cdot U(t) \]

Find:
(a) \( E(X(t)) \)
(b) \( R(t_1,t_2) \)
(c) \( \text{var } X(t) \)

\[ \begin{align*}
(a) \quad E[X(t)] &= E[P \cdot e^{-Qt} \cdot U(t)] \\
&= E[P] \cdot E[e^{-Qt} \cdot U(t)] \quad ; \quad E[P] = \frac{1}{2} \\
&= E[e^{-Qt}] \\
&= \int_0^1 e^{-qt} \, dq = -\int e^{-qt} \, \Big|_0^t = \frac{1-e^{-t}}{t} \\
&\Rightarrow E[X(t)] = \frac{1-e^{-t}}{2t} \cdot U(t) \\
(b) \quad R(t_1,t_2) &= E[X(t_1) \cdot X(t_2)] = E[P^2 \cdot e^{-Q(t_1+t_2)}] \cdot U(t_1) \cdot U(t_2) \\
&= E[P^2] \cdot E[e^{-Q(t_1+t_2)}] \cdot U(t_1) \cdot U(t_2) \\
&= E[P^2] \\
&= \int_0^1 p^2 \, dp = \frac{1}{3} \\
&\quad E[e^{-Q(t_1+t_2)}] = \frac{1-e^{-(t_1+t_2)}}{t_1+t_2} \\
&\Rightarrow R(t_1,t_2) = \frac{1-e^{-(t_1+t_2)}}{3(t_1+t_2)} \cdot U(t_1) \cdot U(t_2) \\
(c) \quad E[X^2(t)] &= R(t,t) = \frac{1-e^{-2t}}{6t} \cdot U(t) \\
\text{var } X(t) &= E[X^2] - E(X)^2 \\
&= \left( \frac{1-e^{-2t}}{6t} - \left( \frac{1-e^{-t}}{2t} \right)^2 \right) \cdot U(t)
\end{align*} \]
We draw \( N \) iid samples from a shifted Laplacian random variable with mean \( \mu \) and variance \( \sigma^2 \). Give an approximation of the density function for the average of these numbers. \( N \gg 1 \)

**Central limit theorem:**

\[
\overline{X} = \frac{1}{N} \sum_{n=1}^{N} X_n
\]

\[
E[\overline{X}] = E[X] = \mu
\]

\[
\text{Var}(\overline{X}) = \frac{\sigma^2}{N}
\]

\[\Rightarrow \overline{X} \text{ in normal } (\text{mean} = \mu, \text{variance} = \frac{\sigma^2}{N})\]

\[
\overline{X} \sim \frac{\sqrt{N}}{\sqrt{2\pi}\sigma} e^{-\frac{(X - \mu)^2}{2\sigma^2/N}}
\]
X(t) is a stationary random process with mean $\mu$ and autocorrelation

$$R(\tau) = \frac{x^2}{\pi} \exp \left( -a|\tau| \right)$$

where $a$ is a specified parameter. What percentage of the time can we expect $X(t)$ to lie below a given threshold, $T$?

$$P_r [X(t) \leq T] = \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi \text{var}}} e^{-\frac{(x-\mu)^2}{2\text{var}}} dx$$

Let $y = \frac{x-\mu}{\sqrt{\text{var}}}$

$$P_r [X(t) \leq T] = \int_{-\infty}^{\frac{T-\mu}{\sqrt{\text{var}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{2} + \text{erf} \left( \frac{T-\mu}{\sqrt{\text{var}}} \right)$$

$$= \frac{1}{2} + \text{erf} \left( \frac{T-\mu}{\sqrt{x^2 - \mu^2}} \right)$$
EE505 Final Examination

Robert J. Marks II

August 20, 1997; 2:20 to 4:20 PM

The examination is closed book and closed notes. No calculators are allowed. Each student is allowed three sheets of notes. All problems are weighted equally. Work must be done in ink.

All work will be done in a test booklet. No scratch paper is needed.

"And I trust that you will discover that we have not failed the test.,; 2 Corinthians 13:6 (English-NIV)

1. Let $X$ and $Y$ be independent random variables and let $Z = X + Y$. Prove or disprove the following general propositions.

(a) $Z = X + Y$  $\iff$  $Z = X + Y$  $\iff$  TRUE

(b) $Z^2 = X^2 + Y^2$

(c) $\text{var}(Z) = \text{var}(X) + \text{var}(Y)$.

(d) $\text{var}(aZ) = a^2 \text{var}(Z)$.

\[
\text{var}(aZ) = (aZ)^2 - (aZ)^2 = a^2 Z^2 - a^2 Z^2 = a^2 \text{var} Z
\]

\[
\Rightarrow \text{var} Z = \text{var} X + \text{var} Y
\]
2.

\[ Y = \frac{1}{N} \sum_{k=1}^{N} X_k^2 \]

where the \( X_k \)'s are i.i.d. random variables with probability density function

\[ f_X(x) = e^{-x}U(x) \]

Estimate the probability density function for the random variable \( Y \) when \( N \) is large.\(^1\)

\[ \overline{Z}_k = \overline{X_k}^2 \]
\[ \frac{1}{N} \overline{Z}_k = \frac{1}{N} \overline{X_k}^2 = 2! = 2 \]
\[ \frac{1}{N} \overline{Z}_k = \frac{1}{N} \overline{X_k}^4 = 4! = 24 \]
\[ \implies \text{var} \overline{Z}_k = 24 - 4 = 20 \]
\[ W = \sum_{k=0}^{N} \overline{Z}_k \implies \overline{W} = N \overline{Z}_k = 2N \]
\[ \text{var} W = N \text{var} \overline{Z} = 20N \]

By c.L.T. \( W \sim n \left( 2N, \sqrt{20N} \right) \)

\[ \frac{W}{N} \sim n \left( 2, \sqrt{\frac{\text{var} W}{N^2}} \right) \]

From Prob. 1d

\[ = n \left( 2, \sqrt{\frac{20}{N}} \right) \]

\[ = \frac{1}{\sqrt{2\pi \frac{20}{N}}} \cdot e^{-\frac{(y - 2)^2}{2 \left( \frac{20}{N} \right)}} \]

\[ = \frac{1}{\sqrt{40\pi \frac{N}{4}}} \cdot e^{-\frac{N(y - 2)^2}{40}} \]

\(^1\) Recall from the last test that the \( n \)th moment of each \( X_k \) is \( \frac{1}{n!} \).
A total of \( N \) i.i.d. Bernoulli trials with probability of success \( p \) are performed. The outcome of trial \( m \), the random variable \( X_m \), is set to one if there is a success and zero otherwise. We form the sum

\[
Y = \sum_{m=1}^{N} X_m.
\]

Evaluate the exact probability density function for the random variable \( Y \).

This is a disguised binomial R.V.

\[
p_k = P_r[Y = k] = \binom{N}{k} p^k q^{N-k}; \quad q = 1 - p
\]

\[
f_Y(x) = \sum_k p_k \delta(x - k)
\]

\[=
\sum_{k=0}^{N} \binom{N}{k} p^k q^{N-1} \delta(x - k)\]
4. The Weibull random variable $Y$ with positive parameters $A$ and $B$ is

$$F_Y(y) = \left[1 - \exp\left(-\frac{y}{A}\right)^B\right] U(y).$$

Let $X$ be a uniform random variable on the interval $(0, 1)$. Given $A$ and $B$, find a random variable transformation, $Y = g(X)$, to produce a Weibull random variable.

$$x = 1 - e^{-\left(\frac{y}{A}\right)^B}$$

$$e^{-\left(\frac{y}{A}\right)^B} = 1 - x$$

$$\left(\frac{y}{A}\right)^B = -\ln(1 - x)$$

$$y = A \left[-\ln(1 - x)\right]^{1/B}$$

Thus

$$Y = g(X)$$

where

$$g(x) = A \left[-\ln(1 - x)\right]^{1/B}$$

This is a strictly increasing solution. We can show a strictly decreasing solution:

$$g(x) = A \left[-\ln(x)\right]^{1/B}$$

Are the other $g(x)$'s that will work? Of course!
5. A random variable has a probability density function of

\[ f_X(x) = e^{-x}U(x) \]

We take i.i.d. samples from this distribution until we get a number between zero and one - and then stop. Call this last random variable \( Y \). Evaluate the probability density function of \( Y \).

\[
f_Y(y) = \Pr \left( \frac{X}{X} \leq y \right) = f_X \left( x \mid 0 \leq x \leq 1 \right)
\]

Thus

\[
f_Y(x) = \begin{cases} \frac{e^{-x}}{1-e^{-1}} & ; 0 \leq x \leq 1 \\ 0 & ; \text{otherwise} \end{cases}
\]
6. Let $X$ be a zero mean normal random variable with variance $\sigma^2$. Let $Y = X$ when $X$ is positive and let $Y = 1$ otherwise. Evaluate and sketch the probability density function for $Y$.

By inspection:

$$f_Y(y) = \frac{1}{2} \delta(y-1) + \frac{1}{2} f_X(y) \mathcal{U}(y)$$

$$= \frac{1}{2} \delta(y-1) + \frac{1}{2 \sqrt{2\pi} \sigma} e^{-\frac{y^2}{2\sigma^2}} \mathcal{U}(y)$$

[Sketch of the probability density function]
7. A joint probability density function is defined by 
\[ f_{XY}(x, y) = \begin{cases} A & |y| \leq e^{-x} \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

(a) Evaluate \( A \).
(b) Evaluate the marginal distribution \( f_Y(y) \).

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \, dy = 1 \]
\[ = A \int_{x=0}^{\infty} \int_{y=-e^{-x}}^{e^{-x}} \, dy \, dx = A \int_{x=0}^{\infty} 2e^{-x} \, dx = 2A \]
\[ \Rightarrow A = \frac{1}{2} \]
\[ y = \pm e^{-x} \]
\[ x = \pm \ln y \]

(b) \[ f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \]
\[ = \frac{1}{2} \int_{0}^{\ln|y|} \, dx = \frac{1}{2} \ln|y| \quad \text{if } |y| < 1 \]
\[ = 0 \quad \text{otherwise} \]
8. Does there exist a zero mean random variable where the Tchebycheff inequality is met? In other words, is there a probability density function for which, for all $k \geq 1$,

$$P(|X| \geq k\sigma) = \frac{1}{k^2} \ ?$$

If so, please specify $F_X(x)$. If not, please explain why.

There are a number of ways to show the answer is "no."

**Way #1:** $P_r[|X| \geq k\sigma] = \frac{1}{k^2} = \frac{1}{2}\left(F_X(k\sigma) - F_X(-k\sigma)\right); \ k \geq 1$

Differentiate with respect to $k$

$$\frac{-2}{k^3} = \sigma f_X(k\sigma) - \sigma f_X(-k\sigma); \ k \geq 1$$

$$\Rightarrow f_X(k\sigma) + f_X(-k\sigma) = \frac{2\sigma^2}{k^3}; \ k \geq 1$$

Set $X = k\sigma$\n
$$\Rightarrow f_X(x) + f_X(-x) = \frac{2\sigma^2}{x^3}; \ x \geq 0$$

Since $\overline{X} = 0$, $\overline{X^2} = \text{var} = \sigma^2 = \int_{-\infty}^{\infty} x^2 f_X(x) dx$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 \left[f_X(x) + f_X(-x)\right] dx$$

$$\geq \frac{1}{2} \int_{x=\sigma}^{\infty} x^2 \left[f_X(x) + f_X(-x)\right] dx$$

$$= \frac{1}{2} \int_{x=\sigma}^{\infty} x^2 \frac{2\sigma^2}{x^3} dx = \frac{\sigma^2}{2} \int_{x=\sigma}^{\infty} \frac{1}{x} dx$$

$$= \frac{\sigma^2}{2} \ln x \bigg|_{\sigma}^{\infty} = \infty \Rightarrow \text{violating assumption } \text{var} = \sigma < \infty.$$

**Way #2.** We can rewrite $P_r[|X| \geq \varepsilon] \leq \frac{\sigma^2}{\varepsilon} \Leftrightarrow \text{Tchebycheff's inequality}$

This is the way it is on p.114 of the text. In the derivation, for the $\varepsilon$ integration over the interval $|x| \geq \varepsilon$, we use the inequality $x^2 \geq \varepsilon^2$. From 3rd equation on p.114,

$$\sigma^2 = \int_{|x| \leq \varepsilon} x^2 f_X(x) dx = \varepsilon^2 \int_{|x| \geq \varepsilon} f_X(x) dx$$

Since $x^2$ is changing and $\varepsilon^2$ is not, the inequality should be a strict inequality. Thus, the bound is never met.

There are other ways...
INSTRUCTIONS:

- Rip off the last sheet of paper in this test booklet and put it in your pocket. It is your homework assignment due in class one week from today.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single sheet of notes.
- No calculators pleased.
- Please use an ink pen.
- Each problem is worth 20 points.
- TIE Students: Please ask your proctor to write his/her name on the cover, indicate the times over which the exam was administered and initial.

1. Specify whether the following pairs of events are (a) independent, (b) mutually exclusive (c) both or (d) neither. Two points will be given for a correct answer, zero for no answer and -1 for an incorrect answer.

   - You pass this test. You fail this test. (You may assume the probability of both events is nonzero). Mutually Exclusive
   - A rolled die shows three dots. A flipped coin shows heads.
   - The sum on two dice is seven. There are six dots on the first die. Independent
   - The sum on two dice is seven. There are six dots on one of the two dice. neither
   - You have an ace in your poker hand. Your opponent has an ace in their poker hand. neither
   - You have an ace in your poker hand. Your opponent has the king of hearts in their poker hand. mutually exclusive
   - You win the Washington state lottery. Your mother wins the Ohio State lottery. (Both of you purchased tickets.) Independent
   - You receive one telephone call before noon. You receive two calls all day. neither
   - You roll two conventional six sided dice. The first die shows three dots. The second die shows thirty eight dots. both
2. Ken Griffey Jr. has a batting average of 0.333. Assume this means, each time he bats, his probability of getting a hit is 1/3. Estimate the probability that he gets between 20 and 28 hits (inclusive) in his next 72 at bats.

\[
P_r[k_1 \leq k \leq k_2] = G \left( \frac{k_2 - np}{\sqrt{npq}} \right) - G \left( \frac{k_1 - np}{\sqrt{npq}} \right)
\]

\[
\sqrt{npq} = \sqrt{72 \times \frac{1}{3} \times \frac{2}{3}} = \sqrt{16} = 4
\]

\[
np = \frac{1}{3} \times 72 = 24
\]

\[
\frac{k_2 - np}{\sqrt{npq}} = \frac{28 - 24}{4} = 1
\]

\[
\frac{k_1 - np}{\sqrt{npq}} = \frac{20 - 24}{4} = -1
\]

\[
\Rightarrow P_r[k_1 \leq k \leq k_2] = \text{erf}(1) - \text{erf}(-1)
\]

\[
= 2 \text{erf}(1)
\]

\[
= 2 \times 0.34
\]

\[
= 0.68
\]
3. In the HUB, there are 10 AA, 20 civil engineering and 30 EE students eating husky burgers. Five students are chosen at random. What is the probability that there are exactly three EE’s and exactly one civil engineering student chosen?

If you assume choice with replacement:

\[ p_1 = \frac{10}{60} = \frac{1}{6}, \quad p_2 = \frac{20}{60} = \frac{1}{3}, \quad p_3 = \frac{30}{60} = \frac{1}{2}, \quad n = 5 \]

\[ P_r[k_1, k_2, k_3] = \frac{n!}{k_1! k_2! k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \]

\[ P_r[k_1 = 1, k_2 = 1, k_3 = 3] = \frac{5!}{1!1!3!} \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot (\frac{1}{2})^3 \]

\[ = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} = \frac{5}{36} \]

If you assume no replacement, it’s pretty rough to figure.
4. Bill eats only sushi and sausage pizzas. Sushi give him heartburn 10% of the time. The pizza's give him heartburn 20% of the time. He eats twice as many pizzas as sushi. Bill has heartburn. What is the probability it was caused by a sausage pizza?

\[
2 P_{\text{Pizza}} = P_{\text{Sushi}} \implies P_{\text{Pizza}} = \frac{2}{3}, \quad P_{\text{Sushi}} = \frac{1}{3}
\]

\[
P_r[\text{Pizza}/H] = \frac{P_r[H/\text{Pizza}] P_r[\text{Pizza}]}{P_r[H/\text{Pizza}] P_r[\text{Pizza}]+P_r[H/\text{Sushi}] P_r[\text{Sushi}]}
\]

\[
= \frac{\frac{1}{5} \times \frac{2}{3}}{\frac{1}{5} \times \frac{2}{3} + \frac{1}{10} \times \frac{1}{3}} = \frac{2/15}{4/30 + 3/30} = \frac{2/15}{7/30}
\]

\[
= \frac{2/15}{7/30} = \frac{4/30}{7/30} = \frac{4}{7}
\]
5. Poisson points with parameter $\lambda = 2$ occurrences per hour are observed for a half hour. What is the probability that the number of occurrences exceeds two given that the total number of occurrences exceeds one?

\[
\lambda = \frac{2}{\text{hr}}
\]

\[
\Pr[X > 2 | X > 1] = \frac{\Pr[X > 2, X > 1]}{\Pr[X > 1]} = \frac{\Pr[X > 2]}{\Pr[X > 1]}
\]

\[
= \frac{1 - \Pr[X \leq 2]}{1 - \Pr[X \leq 1]} = \frac{1 - e^{-\lambda} \left[ \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} \right]}{1 - e^{-\lambda} \left[ \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} \right]}
\]

\[
\lambda = 2 \times \frac{1}{2} = 1
\]

\[
= \frac{1 - e^{-\lambda}[1+1+2]}{1 - e^{-\lambda}[1+1]} = \frac{1 - 4e^{-1}}{1 - 2e^{-1}} = 0.560
\]

\[
\text{CAN LEAVE ANSWER THIS WAY.}
\]
6. Consider a Bernoulli trial with probability of success $p$. We perform the Bernoulli trial until we get a success. Let $M$ denote the number of trials needed to achieve a success.

1. What is the probability that, for a given positive integer, $m$, that $M = m$?

2. Do all of the probabilities add to one?\(^1\)

\[ \begin{align*}
1 & \quad 1 \cdot p \\
2 & \quad q \cdot p \\
3 & \quad q^2 \cdot p \\
4 & \quad q^3 \cdot p \\
\vdots \\
m & \quad q^{m-1} \cdot p \\
\Rightarrow & \quad P_r[M = m] = \begin{cases} 
q^{m-1} \cdot p & ; \quad m = 1, 2, \ldots \\
0 & ; \quad \text{otherwise}
\end{cases}
\end{align*} \]

2. Yes. For credit, show this:

\[ P \sum_{m=1}^{\infty} q^{m-1} = P \sum_{n=0}^{\infty} q^n = \frac{P}{1-q} = \frac{P}{P} = 1 \]

\[ P \sum_{m=1}^{\infty} q^{m-1} = P \sum_{n=0}^{\infty} q^n = \frac{P}{1-q} = \frac{P}{P} = 1 \]

\[ \sum_{m=0}^{\infty} a^m = \frac{1}{1-a} \]

\(^1\)Recall the geometric series where, if $|a| < 1$, 
\[ \sum_{m=0}^{\infty} a^m = \frac{1}{1-a} \]
Homework #3
Due in class on July 28.

1. From Papoulis, Chapter 4: Problems 4,6,7,8,9,10,12,13,14,17,18,19,21

2. There are two classes of objects - a right group and a left group. Both are distributed as Gaussian random variables. The left class has a mean of $p_L$ and a standard deviation of $\sigma_L$. The right class has parameters $p_R$ and $\sigma_R$. An element is chosen with equal probability from one of the groups and the result of the experiment is $P = p$. Show that

$$\text{Probability}[\text{the element is from the left class } | P = p] = \frac{1}{1 + \frac{\sigma_L}{\sigma_R} \exp \left[ \frac{1}{2} \left( \frac{\eta_L}{\sigma_L^2} - \frac{\eta_R}{\sigma_R^2} \right) \right]}$$

where

$$\eta_L = (p - p_L)^2$$

and

$$\eta_R = (p - p_R)^2$$

$$P_r[P = p] = \frac{1}{\sqrt{2\pi}\sigma_L} e^{-\frac{(p - p_L)^2}{2\sigma_L^2}} P_r[P = p]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_R} e^{-\frac{(p - p_R)^2}{2\sigma_R^2}} P_r[P = p] + \frac{1}{\sqrt{2\pi}\sigma_L} e^{-\frac{(p - p_L)^2}{2\sigma_L^2}} P_r[P = p]$$

$$= \frac{1}{1 + \frac{\sigma_L}{\sigma_R} e^{\frac{1}{2} \left[ \frac{\eta_L}{\sigma_L^2} - \frac{\eta_R}{\sigma_R^2} \right]}}$$
INSTRUCTIONS:

- Look on the web for the next homework assignment due one week from today.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed two sheets of notes.
- No calculators pleased.
- Please use an ink pen.
- Each problem is worth 20 points.
- TIE Students: Please ask your proctor to write his/her name on the cover, indicate the times over which the exam was administered and initial.

Some potentially helpful equations follow.

\[
\sum_{k=1}^{\infty} \frac{1}{n} = \infty \\
\cos(\theta) = \frac{1}{2} (e^{j\theta} + e^{-j\theta}) \\
\sin(\theta) = \frac{1}{j} (e^{j\theta} - e^{-j\theta})
\]

1. Let \( X \) be a Poisson random variable with parameter \( a = 1 \). Let \( Y = (X - 3)^2 \). Then

\[
\text{Probability } [-1.5 \leq Y < 1.2] = c \times e^{-1}.
\]

Evaluate the constant \( c \).

\[
\begin{align*}
\Pr[Y = 0 \text{ or } X = 1] &= e^{-a} \\
\Pr[X = 3 \text{ or } X = 2 \text{ or } X = 4] &= e^{-a} \left( \frac{a^3}{3!} + \frac{a^2}{2!} + \frac{a^4}{4!} \right) \\
&= e^{-1} \left( \frac{1}{6} + \frac{1}{2} + \frac{1}{24} \right) \\
&= e^{-1} \left( \frac{4 + 12 + 1}{24} \right) = e^{-1} \left( \frac{17}{24} \right) \\
\Rightarrow c &= \frac{17}{24}
\end{align*}
\]
2. Let $X$ be a uniform random variable on the interval $(-1, 1)$. Evaluate a strictly decreasing nonlinearity, $g(x)$ so that the random variable $Y = g(X)$ is distributed as

$$f_Y(z) = 2e^{-2z}U(z)$$

where $U(\cdot)$ is the unit step.

$$f_X(y) = \left| \frac{d g^{-1}}{dy} \right| f_X(g^{-1}(y)) = 2e^{-2y}U(y)$$

$$\frac{1}{2} \left| \frac{d g^{-1}}{dy} \right| = 2e^{-2y} \Rightarrow \frac{d g^{-1}}{dy} = \pm 4e^{-2y}$$

$$g^{-1}(y) = x = \mp 2e^{-2y} + C$$

choose $+$ for strictly decreasing

$$g^{-1}(y) = 2e^{-2y} + C = x$$

$x = 1 \Rightarrow y = 0$

$$\Rightarrow 2 + C = 1 \Rightarrow C = -1$$

$x = g^{-1}(y)$

$$g^{-1}(y) = 2e^{-2y} - 1 = x$$

solve for $y$:

$$2e^{-2y} = x + 1$$

$$e^{-2y} = \frac{x + 1}{2} \Rightarrow -2y = \ln \frac{x + 1}{2}$$

or

$$y = -\frac{1}{2} \ln \frac{x + 1}{2} = g(x)$$
3. Can the function $\cos(\alpha \omega)$ be a characteristic function? If not, why? If so, what is the corresponding probability density function?

Yes

Note:

$\delta(x-a) \leftrightarrow e^{j\alpha \omega}$

Thus, since

$\cos \alpha \omega = \frac{1}{2} [e^{j\alpha \omega} + e^{-j\alpha \omega}]$

$\Rightarrow f_{x}^{}(x) = \frac{1}{2} \delta(x-a) + \frac{1}{2} \delta(x+a) \leftrightarrow \cos \alpha \omega = \Phi_{x}^{}(\omega)$
4. Let $X$ be a uniform random variable on the interval $(0, 1)$. Find the value of the constant $a$ so that

$$E\left(3(X + a)^2\right) = 1.$$

$$\overline{X^n} = \int_0^1 x^n dx = \left. \frac{x^{n+1}}{n+1} \right|_0^1 = \frac{1}{n+1}$$

$$3(\overline{X+a})^2 = 3(\overline{X^2} + 2a \overline{X} + a^2)$$

$$= 3 \left( \frac{1}{3} + 2a \frac{1}{2} + a^2 \right)$$

$$= 1 + 3a + 3a^2 \quad = 1$$

$$\Rightarrow 3a + 3a^2 = 0$$

$$1 + a = 0 \Rightarrow a = -1$$

or, simply integrate

$$3(\overline{X+a})^2 = 3 \int_0^1 (x+a)^2 dx = 1$$
5. Express

\[ E[\cos(2\pi X t)] \]

in terms of real part of the characteristic function of \( X \).

\[ \cos 2\pi X t = \frac{1}{2} e^{j2\pi X t} + \frac{1}{2} e^{-j2\pi X t} \]

\[ = \frac{1}{2} \Phi_X(z \pi t) + \frac{1}{2} \Phi_X(-z \pi t) \]

But:

\[ \Phi_X(-z \pi t) = \int_{-\infty}^{\infty} f_X(x) e^{jz\pi x \omega} dx \]

\[ = \left[ \int_{-\infty}^{\infty} f_X(x) e^{jz\pi x \omega} dx \right]^* \]

\[ = \Phi_X^*(z \pi t) \]

Thus:

\[ \Rightarrow \cos 2\pi X t = \frac{1}{2} \Phi_X(z \pi t) + \frac{1}{2} \Phi_X^*(z \pi t) \]

\[ = \Re \Phi_X(z \pi t) \]
6. Is the following inequality true when the probability density function of $X$ is zero for negative $x$?

$$\text{Probability}[X \geq a] \leq E \left[ \left( \frac{X}{a} \right)^{2n} \right]$$

If so, please show. If not, give a counterexample.

Sure.

Markoff Inequality

$$\text{Pr}[X \geq a] \leq \frac{E}{a}$$

$$E = a^{2n}$$

$$\Rightarrow \text{Pr}[X^{2n} \geq a] \leq \frac{a^{2n}}{a}$$

$$= \text{Pr}[X \geq a^{2n}]$$

set $a = a^{2n}$. This gives

$$\text{Pr}[X \geq a] \leq \frac{a^{2n}}{a^{2n}}$$
EE 505
Midterm

INSTRUCTIONS:

- Monday, July 15, 1996; 2:20 PM to 4:20 PM.
- Write your name on the upper right hand side of this sheet.
- Do all of your work in this test booklet.
- This test is closed book and closed note.
- You are allowed a single legal sized sheet of notes and calculator.
- Each problem is worth 20 points.
- TIE students must identify the exam proctor and have the proctor initial the examination.

1. Specify whether the following pairs of events are (a) independent, (b) mutually exclusive or (c) neither.

- A balanced budget amendment bill passes congress by August 15. A balanced budget amendment does not pass congress by August 15.
- The sum on two dice is seven. There are six dots on the first die.
- You have an ace in your poker hand. Your opponent has an ace in their poker hand.
- You receive one call before noon. You receive two calls all day.

1 Four points for a correct answer, zero for no answer and -2 for an incorrect answer.
2. Ken Griffey Jr. has a batting average of 0.300. Assume this means, each time he bats, his probability of getting a hit is 0.300. Estimate the probability that he gets 300 or more hits in his next 1000 at bats.
3. In Lake Wobegon, there are 10,000 catfish, 20,000 perch and 30,000 dogfish. The dogfish are very hungry and are twice as likely to be caught. Bobby Jack went fishing and caught four fish. The probability that three were dogfish and one was a perch were caught can be written as

\[ \frac{2^P}{3^Q} \]

What are the integers \( P \) and \( Q \)?
4. Bill eats only Big Macs and sausage pizzas. Big Macs give him heartburn 10% of the time. The pizza’s give him heartburn 20% of the time. He eats twice as many pizzas as Big Macs. Bill has heartburn. What is the probability it was caused by a Big Mac?
5. A Poisson random variable with parameter $\lambda = 2$ occurrences per hour is observed for a half hour. The probability that the number of occurrences exceeds or is equal to two given that the total number of occurrences exceeds or equals one can be written as

$$\frac{1 - a e^c}{1 - b e^d}.$$ 

Identify the numbers $a$, $b$, $c$ and $d$. 

5
6. Washington state apples are modeled with a Gaussian pdf. If \( X \) is the diameter,
\[
    X \sim N(\mu = 3, \sigma = 2)
\]
Apples below two inches in diameter and above four inches are discarded. What is the probability that an apple passing this test is three inches or less in diameter?
7. Matlab's error function is

\[ \text{erf}_{ML}(x) = \frac{2}{\sqrt{\pi}} \int_{t=0}^{x} e^{-t^2} dt \]

Papoulis' definition is

\[ \text{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{z=0}^{y} e^{-\frac{z^2}{2}} dz \]

We wish to find \( \text{erf}(2) \) using Matlab. How do you do it?
Table 3-1 \( \text{erf} \: x = \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-y^2/2} \: dy = G(x) - \frac{1}{2} \)

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<th>( x )</th>
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Papoulis
2nd ed.
H.W.

#1: Chapt 2: 1, 8, 17, 19, 21
#2: Chapt 3: 2, 7, 10, 13
#3: Chapt 3: 3
#4: Chapt 5: 14, 21, 23, 27, 28, 30
#5: Chapt 6: 2, 3, 5
#6: Chapt 7: 11, 10, 12, 15
#7: Chapt 9: 1, 3, 5, 6, 12, 14
#8: Chapt 9: 7, 11, 16, 21, 25

Chapt 10: 3, 5, 7

2 MTS's
1 FINAL
Hw.

#1 Chapt 2: 1, 2, 3, 4, 12, 13
\[ 5 \leq B \leq 7 \]
\[ 5 \leq 2B \leq B + 7 \]