Optimal & Adaptive Control
Texas Tech University (1976)
R.J. Marks II Class Notes
\[ \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \end{bmatrix} \]

\[ x' = \dot{x} = x \]
\[ y' = \dot{y} = y \]
\[ z' = \dot{z} = z \]

\[ \dot{x} = x(t) + u(t) \]

Example:

\[ x(0) = 0, \quad \dot{x}(0) = 0 \]

Performance Measures:

- Physical Constraints
- Output States

\[ x(t) + y(t) = d(t) \]
Theorem: \( A \) and \( M \) allowable control

\[
\begin{align*}
0 \leq x(t) &< M \\
0 \leq x(t) &< M \\
0 \leq x(t) &< M \\
0 \leq x(t) &< M
\end{align*}
\]

Constraints on \( x \) (states)

Constraints on \( u \) (input)

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

When \( A = \text{Matrix} \)
Assume fuel consumption is a constant to neglect. (Rate)

Fuel Consumption Rate:  

\[
\begin{align*}
\int_{t}^{t+\Delta t} (k_1 P(t) + k_2 x(t)) \, dt &= C \quad \text{assumed} \\
\end{align*}
\]

Say you find some practical control \( u = q(t) \)
Property II. Immediate Follows

or we have shown

\[ x(t) = 0 \quad x(t) = 0 \quad x(t) = 0 \]

Case 1: Above two gives

\[ x(t) = 0 \quad x(t) = 0 \quad x(t) = 0 \]

It follows that

\[ x = 0 \quad x = 0 \quad x = 0 \]

Consider then

\[ \begin{pmatrix} \Phi(t,t_0) \end{pmatrix} \begin{pmatrix} x(t) \end{pmatrix} = 0 \quad \begin{pmatrix} x(t) \end{pmatrix} = 0 \quad \begin{pmatrix} x(t) \end{pmatrix} = 0 \]

If \( t = t_0 \) is the fundamental matrix

\[ \Phi(t) = A(t) = \Phi(t) \]

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It satisfies:

\[ \Phi(t_0) = \text{state transition matrix} \]

\[ \Phi(t_0) = \text{state transition matrix} \]

\[ \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{pmatrix} x(t) \end{pmatrix} = \begin{point} −1 \ -7 \ -16 \ (\text{eq}) \end{pmatrix}
\[
\begin{align*}
\mathbf{A} + \mathbf{B} &= \mathbf{C} \\
\mathbf{A} - \mathbf{I} &= \mathbf{B} \\
\mathbf{C} &= \mathbf{D} \\
\mathbf{E} &= \mathbf{F}
\end{align*}
\]
which is closed form

\[ P = (2) \phi \]

Then is a diagonalized matrix

\[ P = e^{t}p \]

\[ \text{Here, } A = PDP^{-1} \]

\[ \text{Find } A \]

An alternative method

\[ (0) \phi (t) = P^{-1} (ST - A) P \]
(\psi(t), 0, \varphi(0) = 0) \quad \text{with} \quad A \text{ bounded}

\text{In any finite time}

\text{to any final state } x_f

\text{state } x(t_0) \text{ can be expressed if } t_0 \text{ to } t, \text{ each initial complete state is attainable. A system is said to be completely state-controllable if the system in cont.}

\text{To a desired final state } x_f \text{ one can choose } u \text{ to move}

x = f(x(t), u(t)), \quad x(t_0) = x_0

\text{Consider control of systems of the}

\text{state variable representation}
\[
\begin{align*}
\text{Claim:} & \quad x = 0 \text{ if } y \neq 0 \\
\text{Proof:} & \quad y(t) = t^2 + 1 \\
\text{LHS:} & \quad [t_0, t_1] \\
\text{RHS:} & \quad [t_0, t_1] \\
\text{Thus, } y \text{ is not singular.} \\
\end{align*}
\]
\[ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = u \]

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} x' \\ y' \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\end{align*}
\]

\[
\begin{align*}
C W C^T &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} C^T \\
W &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} C^T
\end{align*}
\]

For some \( W \), \( W^T W = I \), and \( C \) is a \( m \times n \) matrix with \( m < n \) and \( C^T C \) is invertible.

Since \( M \) is symmetric, we have

\[ M = M^T \]

\[ C M = 0 \]

Thus \( M \neq 0 \) and the system is stable.

Assume \( M^T \neq 0 \).

Then \( C \in \mathbb{C}^n \setminus \mathbb{C}^m \).

Proof by contradiction (neccessary)

\[ \text{NECESSARY} \]
Thus, we have a contradiction.

\[ C = 0 \]

\[ \frac{d}{dt} \left( t \right) x(t) = 0 \]

Thus, let \( x(t) = C \). Thus, let \( x(t) \) be true. A \( x(t) \) must be true of \( (t) \).

Thus, we have the state controllability.

\[ 0 = C \left[ 0 = \begin{bmatrix} \frac{d}{dt} x(t) & x(t) & 0 \end{bmatrix} = 0 \right] \]

\[ \text{Thus } C \text{ is not positive. Matrix} \]
Proof follows as previously complete.

The system is controllable in the sense that \( \dot{y}(t) = y_0 + \int_{t_0}^{t} \dot{u}(\tau) d\tau \) if

\[
\begin{bmatrix}
\dot{y}(t)
\end{bmatrix} = \begin{bmatrix}
0
\end{bmatrix} + \begin{bmatrix}
1
\end{bmatrix} \begin{bmatrix}
\dot{u}(t)
\end{bmatrix}
\]

Output controllability.
Consider $\mathbf{H}^T(\mathbf{t}^\perp, \mathbf{t}^\perp) \mathbf{H}(\mathbf{t}^\perp, \mathbf{t}^\perp) = 0$ in the linear-algebraic view. Let $\mathbf{x} = \mathbf{A} \mathbf{t}^\perp + \mathbf{b}$. Then $\mathbf{H}(\mathbf{t}^\perp, \mathbf{t}^\perp) \mathbf{x} = \mathbf{f} = \mathbf{c}$. Consider $1 = \mathbf{M}^T(\mathbf{t}^\perp, \mathbf{t}^\perp) \mathbf{M}(\mathbf{t}^\perp, \mathbf{t}^\perp)$. Due to the reasoning of (9.115), we have $\mathbf{f} = \mathbf{c}$. HOMEWORK 2

\[ \frac{418176}{(60)} \]

Now consider $\mathbf{d}^T(\mathbf{e}, \mathbf{f}) = \mathbf{H}(\mathbf{e}, \mathbf{f}) \neq 0$. Or $\mathbf{C}^T(\mathbf{e}, \mathbf{f}) \neq 0 \Rightarrow \mathbf{C} \mathbf{e} \neq \mathbf{0}$.

Then $\mathbf{y} = \mathbf{H}^T(\mathbf{e}, \mathbf{f}) \mathbf{H}(\mathbf{e}, \mathbf{f})$ is non-singular. The column vectors of $\mathbf{H}(\mathbf{e}, \mathbf{f})$ are all linearly independent.
\[
\begin{align*}
\frac{\partial}{\partial t} \frac{\partial}{\partial t} A(t) &= \frac{\partial}{\partial t} A(t) \\
&= A(t) \\
&= A(t)
\end{align*}
\]
In general, we have recursion

\[ L_n (t) = L_{n-1} (t) - A(t) L_{n-2} (t) \]

One more time gives

\[ L_n (t) = L_{n-1} (t) - A(t) L_{n-2} (t) \]

\[ L_{n-1} (t) = L_{n-2} (t) - A(t) L_{n-3} (t) \]

\[ L_{n-2} (t) = L_{n-3} (t) - A(t) L_{n-4} (t) \]

Let \[ L_0 (t) = 1 \] and \[ L_1 (t) = A(t) \]

\[ L_2 (t) = A(t)^2 \]

\[ L_3 (t) = A(t)^3 \]

\[ L_4 (t) = A(t)^4 \]

\[ L_5 (t) = A(t)^5 \]

\[ L_6 (t) = A(t)^6 \]

\[ L_7 (t) = A(t)^7 \]

\[ L_8 (t) = A(t)^8 \]

\[ L_9 (t) = A(t)^9 \]

\[ L_{10} (t) = A(t)^{10} \]

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\[ L_{99} (t) = A(t)^{99} \]

\[ L_{100} (t) = A(t)^{100} \]

Let's now differentiate.
\[ J = \begin{bmatrix} B & A_1 & A_2 & \ldots & A_n \end{bmatrix} \]

For a square matrix, for an input \( \text{rank} = 1 \), the columns of the matrix must be linearly independent. If the rank is \( \text{rank} = n \), the columns are not linearly independent.

The columns of the matrix \( \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_n \end{bmatrix} \) have a determinant of zero, i.e.,

\[ \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_n \end{bmatrix} = 0 \]

Actually, if we carried it out,

\[ \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_n \end{bmatrix} \]
Note: For cont. we want $H(x)$.

Theorem: The system $(\ast)$ is complete.

From the knowledge of $f(t)$ on $[t_0, t_f]$, can be determined uniquely the initial state of the unforced system $x(t_0) = x_0$ for $x(t)$ on $[t_0, t_f]$. To be complete.

Def.: The system $(\ast)$ is said to be

\[
\text{(controllability)}
\]

\[
\text{(observability)}
\]
Let \( L \) be a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \), and let \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) be a basis for \( \mathbb{R}^n \) and \( \mathbf{w}_1, \ldots, \mathbf{w}_m \) be a basis for \( \mathbb{R}^m \). Then the matrix of \( L \) with respect to these bases is defined as:

\[
\begin{bmatrix}
L(\mathbf{v}_1) & L(\mathbf{v}_2) & \cdots & L(\mathbf{v}_n)
\end{bmatrix}
\]

Thus, the system is solvable if and only if \( \det(A) \neq 0 \), where \( A \) is the matrix of the system.

Recall that a system is equivalent to another system if they have the same solutions.

Proof:

As before, by contradiction.

NECESSITY: ESTABLISHED

\( \text{RECALL: LINEARLY IND.}\)

\( \text{IF COHOMOLOGY OF } C(f(t, t')) \):
Notice: control.

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
F = [A, B, C, D] = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
0 & 1
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
0 & 1
\end{bmatrix}
\]

System is not observable. System is linear in columns.

Look at output (output)

Look at observable (state)

Look at controlable (state)

Example: \( (A, \Phi) \)
The system cannot be in line only if \( x = 0 \) or \( x = \frac{5}{2} \).

Two decoupled systems:

\[
\begin{align*}
0 \quad & x^2 + x^2 = -v(t) + (5) - x_1 + x_1 = v(t) \\
0 \quad & x^2 + x^2 = -v(t) + (5) - x_1 + x_1 = v(t) \\
\end{align*}
\]

Substituting this result back:

\[
\begin{align*}
x^2 &= \frac{v(t)}{1 + s} \\
1 + s &= \frac{v(t)}{x^2} \\
1 &= \frac{x^2}{v(t)} - s \\
\end{align*}
\]

This does not appear.

Substituting back:

\[
\begin{align*}
x^2 &= \frac{v(t)}{1 + s} \\
1 + s &= \frac{v(t)}{x^2} \\
1 &= \frac{x^2}{v(t)} - s \\
\end{align*}
\]

Consider:

\[
\begin{align*}
\begin{bmatrix}
1 - \frac{v(t)}{x^2} & \frac{2}{1 + s} \\
\frac{v(t)}{x^2} & \frac{1}{1 + s}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
1 + \frac{v(t)}{x^2} & -\frac{2}{1 + s} \\
\frac{v(t)}{x^2} & -\frac{1}{1 + s}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
1 & -\frac{v(t)}{x^2} \\
0 & 1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
1 & -\frac{v(t)}{x^2} \\
0 & 1
\end{bmatrix}
\end{align*}
\]
I = \int y \left( 0 + \int [x(t) + 1]^2 \, dt \right) + \frac{\int [x(t) + 1]^2 \, dt}{2'}

**Performance Measure**

**Control Problem**

Find an optimal control \( u^*(x,t) \) for an admissable trajectory \( x(t) \in \Gamma(x', u') \).

**End of Chapter 1**

**Not Controllable**

\[
0 = \frac{\partial\bar{u}(s)}{\partial x(s)} = \frac{\partial x(s)}{\partial x(s)} + \frac{\partial x(s)}{\partial u(s)}
\]

**Controlled**

\[
0 = \frac{\partial x(s)}{\partial x(s)} + \frac{\partial x(s)}{\partial u(s)}
\]

Let's look at the output.
\[
J = \int_0^T \lVert x(t) - r(t) \rVert dt
\]

Consider a rockey.

### Minimum Fuel Problem

**Minimum Control Effort Problem**

H \text{ weights, controls} \text{ components}

H - sym, pos. semidefinite matrix

\[
J = \int_0^T \lVert x(t) - r(t) \rVert dt
\]

or, we have:

\[
\frac{\partial J}{\partial x(t)} = r(t)
\]

\[
J = \int_0^T \lVert x(t) - r(t) \rVert dt
\]

or final state

\[
0 = r(f)
\]

Concerned

### Terminal Control Problem

\[
J = \int_0^T \lVert x(t) - r(t) \rVert dt
\]

Go from \(x(t_0)\) to \(x(t_f)\) in \(T\) min. time

### Some Special Cases
The most straightforward way to prove this is through the use of the finite-dimensional case, which leads to the following:

\[ \| x(t_1) - x(t_2) \| \leq C \Phi(\tau) \]

Moreover, we can use this result to conclude that the solution is indeed unique and stable.

In energy constraints, we also have constant energy.

Problem: \[ \int_0^T f(t) \geq 0 \]

Special case: relaxation

\( g(t) \) is continuous, \( g(t) \) and \( g(t) + \alpha \) are positive.

\[ \int_0^T g(t) \geq 0 \]

Closest possible C is positive.

Keep \( f(t) \) the desired state.

(a) The time problem

(b) The time problem

\[ \int_0^T f(t) \geq 0 \]

\( g(t) \) is nonnegative.

Minimum control energy

\( \int_0^T f(t) \geq 0 \)
For some solutions, see Appendix Page 170.

\[ v = \sqrt{2a(x^2 + y^2 + z^2)} \]

Let \( g = 0 \) and

\[ \frac{dx}{dt} = \frac{\sqrt{2a}}{v} \]

Then

\[ \frac{dx}{dt} = \frac{\sqrt{2a}}{v} \]

Now, consider the vector function of time. Let

\[ X = X(t) \]

In the plane of \( x \) and \( y \), let

\[ N = N(t) \]

The figure of \( g \) is

Example:
\[ I = \int 0 = \int \frac{2x^2}{x^2 + 1} \]

\[ \therefore (x^2 + 1)^2 = 2x^2 \]

\[ 0 = \frac{2x^2}{x^2 + 1} \]

\[ 0 = 2x^2 \]

\[ 0 = \frac{2x^2}{x^2 + 1} \]

\[ \exists (x) \]

Derivatives to zero.

Select variables. Set to zero.

What about a function of x?

\[ \Rightarrow 0 = 0 \]

\[ 20 = 20 \]

\[ \Rightarrow \]

\[ 0 = \frac{x^2}{x^2 + 1} \]

\[ \exists M \]

\[ \Rightarrow 0 < 0 \]

\[ 0 < 0 \]

\[ \Rightarrow 0 > 0 \]

\[ 0 < 0 \]

\[ \Rightarrow 0 > 0 \]

\[ \Rightarrow 0 = 0 \]

\[ \Rightarrow 0 = 0 \]

\[ \Rightarrow 0 = 0 \]
\[
\begin{align*}
\text{M.N} & = \begin{bmatrix} 1/2 \\ -1 \\ -1 \end{bmatrix} \\
\text{CALLED EXTREMA}
\end{align*}
\]

\[
\begin{align*}
x_1, x_2 & \geq 0 \\
1 \cdot x_1 + 1 \cdot x_2 & \geq 2 \\
(1) \text{ EXTREMA}(x) \leq
\end{align*}
\]

\[
\begin{align*}
(1-x_1, 2-x_2, x_3 & ) \geq 0 \\
\text{FAMILY OF CONICS} & \quad \text{GA}
\end{align*}
\]

\[
\begin{align*}
0 & = (x-1)^2 + (x^2-1)^2 - \frac{1}{2} \\
\text{GA} & = \frac{1}{2} \cdot (x-1)^2 + \frac{1}{2} \\
\text{CONSIDER} & = C = \text{CONST} \\
\end{align*}
\]
\( P = \frac{1}{2} \left( x^2 + y^2 \right) \cdot \frac{1}{2} \leq 0 \)

\[ H = p (x^2 + y^2) \]

The solutions:

\[ x = \frac{2}{3} \]

\[ y = \frac{2}{3} \]

Introducing Lagrangian multipliers:

\[ \lambda = 1, 2, \ldots, m \]

\[ x = 0 \]

\[ y = 0 \]

\[ \lambda = 1, 2, \ldots, m \]

Also:

\[ \frac{d}{dt} \mathcal{L} = 0 \]

Consider the first-order conditions:

\[ \frac{\partial \mathcal{L}}{\partial x} = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial y} = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda} = 0 \]

In general, Lagrange Multipliers Theorem - Constraints & Extrema Problems.

\[ y(x, y) = \frac{1}{2} (x^2 + y^2) \]

\[ \lambda = 1, 2, \ldots, m \]

\[ \min_{x, y} f(x, y) + \sum_{i=1}^{m} \lambda_i g_i(x, y) \]

\[ \text{subject to } g_i(x, y) = 0, i = 1, \ldots, m \]
unknown

Recall, \( f(x) = 0 \) \( \forall x \in \mathbb{R} \)

Case 1: \( x \leq \frac{1}{2} \)

Case 2: \( x > \frac{1}{2} \)

Thus, there will must be

All terms are linearly independent.

\[
0 = \left( x_1 + x_2 + \ldots + x_m \right) + \left( x_1 + \frac{x_2}{2} + \ldots + \frac{x_m}{2^{m-1}} \right) + \ldots + \left( x_1 + \frac{x_2}{2} + \ldots + \frac{x_m}{2^{m-1}} \right)
\]

Combining variables

\[
0 = \left( x_1 + x_2 + \ldots + x_m \right) + \left( x_1 + \frac{x_2}{2} + \ldots + \frac{x_m}{2^{m-1}} \right) + \ldots + \left( x_1 + \frac{x_2}{2} + \ldots + \frac{x_m}{2^{m-1}} \right)
\]

or
\[ \frac{\theta}{\pi} = 2 \implies \theta = 2\pi \]

\[ A = \sqrt{2A_0} \]

plus it into 2 gives

\[ r = \frac{1}{\sqrt{A_0}} \]

\[ V = \frac{2}{A_0} - 2\pi r^2 = 0 \]

\[ \frac{dV}{dr} = 0 \quad \text{gives maxima} \]

\[ V(r) = 2\pi r^2 \left( \frac{A_0}{2\pi} - r^2 \right) \]

\[ f(x) = \pi r^2 + 2\pi r - A_0 = 0 \]

\[ A = 2\pi r^2 + 2\pi r - A_0 \quad \text{constant} \]

\[ V(r) = \pi r^2 \]

\[ \text{Example:} \]

\[ \text{For a given surface,} \]

\[ \text{volume maximize:} \]

\[ r \]

\[ \text{Calculate} \]

\[ \text{h} \]
Using Lagrange Multipliers

\[ f(x,y) = x + y \]

\[ \begin{align*}
  x &= \frac{\partial f}{\partial \lambda} \\
  y &= \frac{\partial f}{\partial \lambda}
\end{align*} \]

\[ \begin{align*}
  x &= \frac{\partial g}{\partial \lambda} \\
  y &= \frac{\partial g}{\partial \lambda}
\end{align*} \]

\[ g(x,y) = x^2 + y^2 - 1 = 0 \]

\[ \Rightarrow \begin{align*}
  x &= \frac{1}{2} \\
  y &= \frac{1}{2}
\end{align*} \]

From (1), \( A = -\frac{1}{2} \)

Thus, \( A = \frac{1}{2} \)

\[ \Rightarrow \begin{align*}
  x &= 0 \\
  y &= \frac{1}{2}
\end{align*} \]

The eq. of knowns \( f(x,y,z) = 0 \)

\[ \Rightarrow \begin{align*}
  x &= 0 \\
  y &= \frac{1}{2}
\end{align*} \]
\[ T = \lambda_0 + \lambda_1 T \]

Need more multipliers

Typed Don WORK

\[ \frac{1}{2} x_1^2 + x_2 = 0 \]

\[ 1 = x_1 + x_2 \]

\[ y = T + y \]

\[ y(x) = x, x \in \mathbb{R} \]

Exam Please!
Then $J_{uv} = 2\pi r^2 dr + 2\pi r^2 d\theta = 2\pi r^2 (dr + d\theta)$

Also: we want $\theta = \frac{d}{dr} (\frac{2\pi r^2}{2}) = 0$

$\delta = 0$, $r = \frac{8}{\sqrt{2}}$

subject to $V = \frac{A}{r} = 0$

Maximize $A$

$\pi r^2 + 2\pi r^2 = A$

$A = \pi r^2 + 2\pi r^2 = A$

Back to tin can problem.

Back to tin can problem.

For extremum

$\lambda = \frac{d}{d\theta} (\theta) + \frac{d}{d\theta} (\theta)$

$\lambda = 0$

$\frac{d}{d\theta} (\theta) + \frac{d}{d\theta} (\theta) = 0$

$\theta = \frac{\pi}{2}$

$\theta$ are equivalent.
SAME RELATION. WE GET LAST TIME, WE MAY USE THIS LAGRANGIAN MULTIPLIERS. WHEN

\[ m = n - 1, \quad z = 1 = 1 \]

\[ \det \begin{bmatrix} \frac{2T}{r} & \frac{2T}{r^2} \\ \frac{2T}{r^2} & \frac{2T}{r^3} \end{bmatrix} \cdot 1 = 0 \]
\[ f(0,0) = 0 \]

\[ f(x,0) = 0 \]

\[ f(0,y) = 0 \]

\[ f(x,y) = x^2 + y^2 \]

\[ f(x,0) = x^2 \]

\[ f(0,y) = y^2 \]

\[ f(x,y) = x^2 + y^2 \]

\[ \text{Initial Conditions} \]

\[ f(x,0) = x^2 \]

\[ f(0,y) = y^2 \]

\[ f(x,y) = x^2 + y^2 \]

\[ \text{Boundary Conditions} \]

\[ f(x,0) = x^2 \]

\[ f(0,y) = y^2 \]

\[ f(x,y) = x^2 + y^2 \]

\[ \text{Solution} \]

\[ f(x,y) = x^2 + y^2 \]

\[ \text{Partial Differential Equation} \]
In general, solution of \( n-1 \) equations in \( n \) unknowns gives us a two-dimensional line in an \( n \)-dimensional space. Note for given initial conditions:
\[
f_k = 0 \quad \forall \ k = 1, 2, \ldots, n-1
\]

But, now we gotta find \( f_0(0) \).

To find \( f_0 = f_0(0) e^{\pm t} \)

\[
x = f_1, f_2 = f_2(0) e^{\pm t}
\]

This gives us:

\[
\frac{\frac{\partial}{\partial x}}{\partial x} + \frac{\partial}{\partial t} \frac{\partial x^2}{\partial x} + \ldots + \frac{\partial^2}{\partial x \partial t} = -f
\]

\[
\frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial t} f_2 + \frac{\partial^2}{\partial x \partial t} = f
\]
Also, we gotta have

\[
\begin{align*}
0 &= y + \frac{x}{3} + \frac{x}{3} \\
\frac{\partial f}{\partial x} &= \frac{x}{3} + \frac{x}{3}
\end{align*}
\]

Now to calculate multiplicity theorem

Assume \( f : u \to \mathbb{R} \) vector

\begin{align*}
x \in \mathbb{R}^n \Rightarrow u & \in \mathbb{R}^n \\
(f(x),u) &= 0 \quad (f \text{ exterior form})
\end{align*}

Remove chart \( \varphi : U \to \mathbb{R}^n \)
\[
\begin{align*}
\left[ \frac{\theta}{2} \right] + 6\times \frac{\theta}{3} \times \frac{8}{3} = 5 \\
= 6 \times \frac{\theta}{3} \\
\end{align*}
\]

\[
\begin{align*}
\frac{6\times (\theta - \frac{\theta}{2})}{3} + \\
\times 5 \left[ \frac{\theta}{3} \right] + \\
(\bar{P}) \bar{P} = \theta \\
\end{align*}
\]

\[(\text{Find second variation})
\]

\[\text{Diagonalize L} \]

\[
\begin{align*}
\bar{V} = \left( \frac{\theta}{3} \right) + 6 \times \frac{\theta}{3} + \left( \frac{\theta}{3} \right)
\end{align*}
\]
\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix}
\begin{pmatrix}
(\frac{\sqrt{3}}{2})x_{11} - \frac{1}{2}x_{12} \\
\frac{1}{2}x_{11} + \frac{1}{2}x_{12}
\end{pmatrix}
+ \begin{pmatrix}
0.5x_{11} - \frac{1}{2}x_{12} \\
\frac{1}{2}x_{11} + \frac{1}{2}x_{12}
\end{pmatrix}
+ \begin{pmatrix}
-\frac{\sqrt{3}}{2}x_{11} - \frac{1}{2}x_{12} \\
\frac{1}{2}x_{11} + \frac{1}{2}x_{12}
\end{pmatrix}
\end{bmatrix}, \\
\mathbf{b} &= \begin{bmatrix}
0.5x_{11} - \frac{1}{2}x_{12} \\
\frac{1}{2}x_{11} + \frac{1}{2}x_{12}
\end{bmatrix}, \\
\mathbf{c} &= \begin{bmatrix}
0.5x_{11} - \frac{1}{2}x_{12} \\
\frac{1}{2}x_{11} + \frac{1}{2}x_{12}
\end{bmatrix}
\end{align*}
\]
Thus, we got the solution

\[ \begin{align*}
\begin{cases}
4x + 8y + 0z &= 0 \\
6x + 3y + 0z &= 0 \\
8x + 4y + 0z &= 0 \\
\end{cases}
\end{align*} \]

\[ \Rightarrow \begin{align*}
4x + 4y + 0z &= 0 \\
6x + 3y + 0z &= 0 \\
8x + 4y + 0z &= 0 \\
\end{align*} \]

\[ \begin{align*}
x &= \frac{3}{2} \left( x + y + z \right) \\
7x &= \frac{3}{2} \left( x + y + z \right) \\
\end{align*} \]

\[ \begin{align*}
x &= 0 \\
y &= 0 \\
z &= 0 \\
\end{align*} \]

Now, let's consider the other cases:

- If \( x + y + z = 0 \)
- If \( x + y + z \neq 0 \)

For linear systems:

\[ \begin{align*}
0 &= \frac{1}{2} \left( x + y + z \right) \\
0 &= \frac{1}{2} \left( x + 2y + z \right) \\
0 &= \frac{1}{2} \left( x + y + 2z \right) \\
\end{align*} \]

Example: Extreme values
\[
\begin{align*}
A &= \begin{bmatrix}
3 & 2 \\
2 & 5
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \\
&= \begin{bmatrix}
4 & 2 \\
2 & 6
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{Let } u &= -R - A \\
\text{Substitute into these into (1).}
\end{align*}
\]

Let's solve for \( x \) and \( y \).
If \( r \) is minimum

\[
x = (H^T \cdot H)^{-1} \cdot H^T \cdot r
\]

\[
\Rightarrow x + H^T \cdot r = \text{least MSE}
\]

\[
0 = -A^T \cdot r (z - H^T \cdot x) = 0
\]

\[
= -A^T \cdot r (z - H^T \cdot x) = 0
\]

\[
= A^T \cdot x (z - H^T \cdot x) = 0
\]

\[
= \frac{1}{2} \| z - H^T \cdot x \|^2 = \min
\]

Find best estimate of \( x \).

\[
\begin{align*}
A & = m \times n \text{ matrix} \\
H & = m \times n \text{ vector} \\
x & = n \times 1 \text{ vector} \\
\end{align*}
\]
\[ X = \text{Average of } Z_i = \frac{1}{m} \sum_{i=1}^{m} Z_i \]

<table>
<thead>
<tr>
<th>[ Z_i ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ 1 ]</td>
</tr>
<tr>
<td>[ 0 ]</td>
</tr>
<tr>
<td>[ 0 ]</td>
</tr>
<tr>
<td>[ 0 ]</td>
</tr>
</tbody>
</table>

A basic OX estimate.

Let \( b = 1, 2 \) be fixed.

\[
(0) \quad \begin{array}{c|c}
Z_i & x = y \\
\hline
1 & 2 \\
2 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\]

\( m \) estimates of a scalar.
\[ \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} + x \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{v} + \mathbf{w} \\ \mathbf{v} \cdot x \end{bmatrix} \]

For our measurement estimation, we have:

\[ x_{\text{bias}} = \frac{1}{n+1} \sum_{i=1}^{n} \mathbf{z}_i \]

\[ x_{\text{new}} = x_{\text{bias}} \]
\[
\begin{bmatrix}
1 + \frac{w_0}{2} & i \frac{h}{2} \\
-\frac{h}{2} & 1 - \frac{w_0}{2}
\end{bmatrix}
\begin{bmatrix}
y \\
-\frac{h}{2}
\end{bmatrix} = \begin{bmatrix} x \\
-\frac{h}{2}
\end{bmatrix}
\]
\[ X_{\text{new}} = X + \Delta X \]

\[ Y_{\text{new}} = Y + \Delta Y \]

\[ Z_{\text{new}} = Z + \Delta Z \]

\[ \Delta X = \frac{\partial X}{\partial p} \Delta p + \frac{\partial X}{\partial h} \Delta h \]

\[ \Delta Y = \frac{\partial Y}{\partial p} \Delta p + \frac{\partial Y}{\partial h} \Delta h \]

\[ \Delta Z = \frac{\partial Z}{\partial p} \Delta p + \frac{\partial Z}{\partial h} \Delta h \]

\[ \Delta \text{Scalar} = \frac{\partial \text{Scalar}}{\partial p} \Delta p + \frac{\partial \text{Scalar}}{\partial h} \Delta h \]

\[ \Delta \text{Matrix} = \Delta \text{Matrix} \]

\[ \Delta \text{Vector} = \Delta \text{Vector} \]

\[ \Delta \text{Scalar} = \Delta \text{Scalar} \]

\[ \Delta \text{Matrix} = \Delta \text{Matrix} \]

\[ \Delta \text{Vector} = \Delta \text{Vector} \]

.. Now...

\[ X_{\text{new}} = \{ \epsilon + h \} \]

\[ \frac{\partial X}{\partial p} = \frac{\partial \text{Scalar}}{\partial p} + \frac{\partial \text{Matrix}}{\partial p} \]

\[ \frac{\partial Y}{\partial p} = \frac{\partial \text{Scalar}}{\partial p} + \frac{\partial \text{Matrix}}{\partial p} \]

\[ \frac{\partial Z}{\partial p} = \frac{\partial \text{Scalar}}{\partial p} + \frac{\partial \text{Matrix}}{\partial p} \]

\[ \frac{\partial \text{Scalar}}{\partial p} = \frac{\partial \text{Scalar}}{\partial p} \]

\[ \frac{\partial \text{Matrix}}{\partial p} = \frac{\partial \text{Matrix}}{\partial p} \]

\[ \frac{\partial \text{Vector}}{\partial p} = \frac{\partial \text{Vector}}{\partial p} \]
\[
\phi(x) \left( \frac{x^2}{2} \frac{1}{x^2} + \frac{3}{x^2} \frac{1}{x^2} \right) = \int_{x_1}^{x_2} \frac{d}{dx} \left( x \phi(x) \right) dx = 0
\]

\[
\phi(x) = \frac{1}{x^2} + \frac{3}{x^2} \frac{1}{x^2} + \frac{2}{x^2} \frac{1}{x^2}
\]

\[
x \phi = \frac{1}{2} \int_{x_1}^{x_2} \frac{d}{dx} \left( x \phi(x) \right) dx = \frac{1}{2}
\]

\[
x \phi = \int_{x_1}^{x_2} \frac{d}{dx} \left( x \phi(x) \right) dx = \frac{1}{2}
\]

**Theorem**

\[
T \phi = \phi(x) + x \phi(x)
\]

\[
\frac{dx}{dt} = \phi(x) + x \phi(x)
\]

\[
\frac{d}{dx} \left( x \phi(x) \right) = \phi(x) + x \phi(x)
\]

**Solution**

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

**Conclusion**

\[
\text{Dynamic Optimization w/ Constraints}
\]

\[
\text{Consider the function}
\]

\[
\frac{d}{dx} \left( x \phi(x) \right) = \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

\[
\phi(x) \rightleftharpoons \phi(x) + x \phi(x)
\]

**The Calculus of Variations**
\[ \frac{\partial F}{\partial y} x^2 - \int x^2 \frac{\partial G}{\partial x} \text{d}x = 0 \]

Then:

\[ \int x^2 \frac{\partial G}{\partial x} \text{d}x = \frac{\partial F}{\partial y} x^2 \]

Consider second term by parts:

\[ \frac{\partial F}{\partial y} = \int x^2 \frac{\partial G}{\partial x} \text{d}x \]

\[ \frac{\partial}{\partial y} [x^2 F(x(t),y(t)) \text{d}x] = \int x^2 \frac{\partial G}{\partial x} \text{d}x \]

\[ e = 0 \quad \implies \quad y = \frac{1}{2} \]

\[ \frac{\partial}{\partial y} = \int x^2 \frac{\partial G}{\partial x} \text{d}x \]

\[ \frac{1}{2} \frac{\partial}{\partial y} = \int x^2 \frac{\partial G}{\partial x} \text{d}x \]

\[ e = 0 \quad \implies \quad y = \frac{1}{2} \]

\[ e = 0 \quad \implies \quad y = \frac{1}{2} \]

\[ \int x^2 \frac{\partial G}{\partial x} \text{d}x = 0 \]

\[ \int x^2 \frac{\partial G}{\partial x} \text{d}x = 0 \]
\text{\textbf{Fixed End Points}}

\text{\textbf{Euler-Lagrange Eqn.}}

\text{\textbf{Transversality Condition}}

\text{\textbf{Conclusions for the Existence of an Extremum}}: 0 - 1, 0 - 2, 0 - 3

\text{\textbf{Conditions}}:

\begin{align*}
\frac{x^3}{3} - \frac{x^5}{5} \bigg|_{0}^{1} &= 0 \\
\frac{\partial}{\partial \gamma} L \bigg|_{\gamma = 0} &= 0 \\
\frac{\partial}{\partial \gamma} L \bigg|_{\gamma = 1} &= 0
\end{align*}

From this, we see two conclusions:
\[ x(t) = 0 \]

\[ y(t) = x(t) \]

**Case 1:** \( x(t) = 0 \)

\[ \frac{\partial}{\partial t} y(t) = 0 \]

\[ \frac{\partial^2 y(t)}{\partial t^2} = 0 \]

**Case 2:** \( x(t) \) is variable

\[ \frac{\partial}{\partial t} y(t) = 0 \]

**Case 3:** \( x(t) \) is fixed

\[ \frac{\partial}{\partial t} y(t) = 0 \]

\[ \frac{\partial^2 y(t)}{\partial t^2} = 0 \]

\[ \frac{\partial^3 y(t)}{\partial t^3} = 0 \]

\[ \frac{\partial^4 y(t)}{\partial t^4} = 0 \]

\[ \frac{\partial^5 y(t)}{\partial t^5} = 0 \]

\[ \frac{\partial^6 y(t)}{\partial t^6} = 0 \]

\[ \frac{\partial^7 y(t)}{\partial t^7} = 0 \]

\[ \frac{\partial^8 y(t)}{\partial t^8} = 0 \]

\[ \frac{\partial^9 y(t)}{\partial t^9} = 0 \]

\[ \frac{\partial^{10} y(t)}{\partial t^{10}} = 0 \]

**Boundary Conditions**

So, the boundary condition is

\[ y(0) = x(0) \]

\[ y'(0) = x'(0) \]

\[ y''(0) = x''(0) \]

To satisfy the boundary conditions,

\[ y(t) = \text{variable functions} \]

\[ y(t) = 0 \]

\[ y(t) = \text{variable functions} \]

\[ y(t) = 0 \]

\[ y(t) = \text{variable functions} \]

\[ y(t) = 0 \]
\[
\frac{\delta}{\delta x} \left[ \frac{1}{2} x^2 \frac{\partial^2}{\partial y^2} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right] = \frac{\partial}{\partial x} \left( \frac{1}{2} x \frac{\partial f}{\partial x} \right)
\]

Sufficient Conditions

Expanding

\[
0 = \frac{s^2}{2} \frac{x}{(s^2 + x^2)^{3/2}}
\]

Euler-Lagrange Eq. 15

F must be twice differentiable
\[ f(x) = \frac{1}{2} [\frac{1}{x^2} + \frac{1}{(x-1)^2}] \]

\[ \int_{0}^{1} f(x) \, dx = \frac{1}{2} \int_{0}^{1} \left( \frac{1}{x^2} + \frac{1}{(x-1)^2} \right) \, dx \]

Second Variation

\[ \frac{\partial^2}{\partial p \partial q} (\phi(q)) = 0 \]

\[ \phi(q) = \int_{0}^{1} [\frac{1}{x^2} + \frac{1}{(x-1)^2}] \, dx \]

\[ = \frac{1}{2} \int_{0}^{1} \left( \frac{1}{x^2} + \frac{1}{(x-1)^2} \right) \, dx \]

\[ = \frac{1}{2} \left( \frac{1}{2} \ln 2 + \frac{1}{2} \ln 2 \right) = \frac{1}{2} \ln 2 \]

\[ \frac{\partial}{\partial p} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3} \]

\[ \frac{\partial}{\partial p} \left( \frac{1}{(x-1)^2} \right) = \frac{2}{(x-1)^3} \]

\[ \frac{\partial^2}{\partial p \partial q} (\phi(q)) = 0 \]

Conclusion of Variation

\[ f(x) = \frac{1}{2} [\frac{1}{x^2} + \frac{1}{(x-1)^2}] \]

\[ = \frac{1}{2} \ln 2 \]
\[ \frac{dx}{dt} = y \quad y = \frac{1}{\sqrt{x}} \]

\[ \frac{dx}{dt} = y \quad y = \frac{1}{\sqrt{x}} \]

\[ x(t) = x(0) e^{\int \frac{1}{u} \, du} = x(0) e^{\frac{t}{2}} \]

\[ x(t) = x(0) e^{\frac{t}{2}} \]

\[ \frac{dy}{dt} = x(0) e^{\frac{t}{2}} \]

\[ \frac{dy}{dt} = x(0) e^{\frac{t}{2}} \]

Consider \( x, y \in \mathbb{R}^+ \)

\[ x^2 + y^2 = \frac{c}{x} \]

\[ x^2 + y^2 = \frac{c}{x} \]

\[ \frac{df}{dt} = \frac{c}{x} \]

\[ \frac{df}{dt} = \frac{c}{x} \]

From Eq. 1, another way to start

\[ \text{Just check derivatives of } \]

\[ \max \text{ or } \min \text{ of } \]

\[ x \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]

\[ x \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]

For \( E = 0 \), \( t = 1 \)

\[ x \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]

\[ x \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]

\[ x \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \]
The functional
Find an extremum of
**Ex:** Both ends are fixed
max
min

\[
x = \frac{\pi \sqrt{\frac{L}{2g}}}{2}
\]

Thus

Thus
Solution is
\[ x(t) = A \cos t + B \sin \left( \frac{t}{2} \right) \]
\[ x(0) = 0 \Rightarrow A = 0 \]
\[ x \left( \frac{\pi}{2} \right) = 1 \Rightarrow B = 1 \text{ and} \]
\[ x(t) = \sin \left( \frac{t}{2} \right) \]

\[ y = x(t) + z \]
\[ x(t) = \sin t + \varepsilon \sin 2t \]

(Note: Boundary conditions check)

\[ J(x) = \int_0^{\pi/2} \left[ \cos^2 \left( \frac{t}{2} \right) - \sin^2 \frac{t}{2} \right] dt \]
\[ = \int_0^{\pi/2} \cos 2t \, dt = 0 \]

\[ J(x + \delta x) = J(x) + \int_0^{\pi/2} \left[ \cos \left( \frac{t}{2} \right) + 2 \varepsilon \cos 2t \right] \delta x \, dt \]
\[ = \frac{3\pi}{4} \varepsilon^2 > 0 \]

\[ J, x = \sin t \text{ is probably a minimum. Let's check} \]
\[ x = \sqrt{\frac{p_2}{p_1}} \quad ( \frac{p_2}{p_1} ) \]

\[ S = \text{Total Length} \]

\[ ds = \sqrt{dx^2 + dy^2} \]

\[ S = \text{Total Length} = \int_{p_1}^{p_2} ds \]

Suppose we can know the line between the point \((x_1, y_1) = 1\) with minimum arc length. We want to find the curve.
Following steps through

\[ 0 = \frac{\partial L}{\partial \dot{x}} = 0 = \text{cons} \]

\[ 0 = \frac{1}{2} m \ddot{x}^2 = m \ddot{x}^2 \]

New Extension

\[ y(0) = 1 = b \]

Boundary condition

\[ y = ax + b \]

\[ y = \frac{y_1}{y_2} \]

\[ y_1 = c_1 (t + y_2) \]

\[ y_2 = c_2 (t + y_2) \]

\[ z = \frac{1}{\sqrt{1 + y_1^2}} \]

Thus

\[ \frac{\partial L}{\partial \dot{z}} = 0 \]

\[ 0 = \frac{ax}{y_2^3} - \frac{a}{y_2^3} \]

Euler Lagrange
\[ \frac{1}{2} \int \left[ \frac{\partial^2}{\partial x^2} \phi(x,t) \right]^2 + \frac{1}{2} \int \left[ \frac{\partial^2}{\partial t^2} \phi(x,t) \right]^2 \, \Omega = 0, \]

Optimal control

\[ \frac{1}{2} \int \left( \frac{\partial^2}{\partial x^2} \phi(x,t) \right)^2 + \frac{1}{2} \int \left( \frac{\partial^2}{\partial t^2} \phi(x,t) \right)^2 \, \Omega = \frac{1}{2} \int \left( \frac{\partial^2}{\partial x^2} \phi(x,t) \right)^2 + \frac{1}{2} \int \left( \frac{\partial^2}{\partial t^2} \phi(x,t) \right)^2 \, \Omega = 0. \]

Assume

\[ \int_0^T \phi(x,t) \, dx + \int_0^T \left( \frac{\partial}{\partial t} \phi(x,t) \right) \, dx = \int_0^T \phi(x,t) \, dx + \int_0^T \left( \frac{\partial}{\partial t} \phi(x,t) \right) \, dx = 0. \]

Assume terminal time problem

\[ \int_0^T \phi(x,t) \, dx + \int_0^T \left( \frac{\partial}{\partial t} \phi(x,t) \right) \, dx = 0. \]
\[ 0 = \frac{\partial}{\partial x} \left[ \frac{x^2 \phi}{\psi} \right] + \frac{\partial^2}{\partial y^2} \phi + \frac{\partial^2}{\partial z^2} \phi + \frac{\partial}{\partial x} \left[ \frac{x^2 \phi}{2 \psi} \right] + \frac{\partial^2}{\partial z^2} \phi + \frac{\partial}{\partial y} \left[ \frac{x^2 \phi}{2 \psi} \right] + \frac{\partial^2}{\partial y^2} \phi \]

\[ = \frac{\partial}{\partial x} \left[ \frac{x^2 \phi}{\psi} \right] + \frac{\partial^2}{\partial y^2} \phi + \frac{\partial^2}{\partial z^2} \phi \]

Simplification:

Before proceeding lets:

\[ \nabla \cdot \phi = 0 \]

\[ \nabla \times \phi = \mathbf{0} \]

\[ \text{NEQ} \rightarrow \text{SET} \rightarrow \text{REL} \rightarrow \text{Relate} \rightarrow \text{Relate} \rightarrow \text{NEQ} \rightarrow \text{ACE} \rightarrow \text{ACE} \rightarrow \text{NOT} \]

Consider the case where
Gives Euler-Lagrange Eq's:
\[
\frac{\partial}{\partial x} \left( \frac{\partial L}{\partial (\frac{dx}{dt})} \right) = \frac{\partial L}{\partial x} \left( \frac{dx}{dt} \right)
\]
\[
L \frac{dx}{dt} \bigg|_{t_0} = 0
\]
This is solution when \( \frac{\partial L}{\partial x} \) and \( \frac{\partial L}{\partial (\frac{dx}{dt})} \) are not related. (ii) For the case where \( t_f \) and \( x(t_f) \) are related by \( x(t_f) = c(t_f) \).

\[ x(t_f) = c(t_f) \]

\[ @ t = t_f \]

\[ x(t_f) + \varepsilon \frac{dz}{dt} x(t_f) = c(t_f) = x(t_f) \]

\[ x[t_f + \varepsilon x(t_f)] + \varepsilon \frac{dz}{dt} x[t_f + \varepsilon x(t_f)] = c(t_f) \]

Differentiate w.r.t. \( \varepsilon \) evaluating result @ \( \varepsilon = 0 \) yields

\[ \frac{d}{dt_f} \frac{dx}{d\varepsilon} + \frac{dz}{dt} x(t_f) + \varepsilon \frac{d}{dt} \left( \frac{dz}{dt} x(t_f) \right) = \frac{sc(t_f)}{dt} \frac{dt}{d\varepsilon} \]

Evaluate @ \( \varepsilon = 0 \)
Example - We wanna minimize

Necessary Conditions

Permanence Terms
Final Time = E.T. = E.T.
Initial Time = E.T. = E.T.

\[ a = \frac{v_f - v_i}{t_f - t_i} \]

\[ x(t) = v_i t + \frac{1}{2} a t^2 \]

\[ \phi = \frac{1}{2} \theta L \]

\[ x = \frac{1}{2} (1 + \cos \theta) \]

\[ a = \frac{v_f - v_i}{t_f - t_i} \]
\[ 1 + 7 = x \]

Also:
\[ x = 0 \implies 1 = (0) \cdot x = 0 \cdot 1 \]

So:
\[ 0 = 1 + \sqrt{\frac{2x - 1}{x(x - 1)}} \]

Taking the square root of both sides:
\[ \sqrt{\frac{2x - 1}{x(x - 1)}} = \sqrt{\frac{2x - 1}{x(x - 1)}} \]

Then:
\[ 0 = \sqrt{x(x - 1)} + \sqrt{\frac{2x - 1}{x(x - 1)}} \]

Thus:
\[ 0 = \sqrt{x(x - 1)} + \sqrt{\frac{2x - 1}{x(x - 1)}} \]

Finally:
\[ 0 = \sqrt{x(x - 1)} + \sqrt{\frac{2x - 1}{x(x - 1)}} \]
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) - \frac{\partial f}{\partial x} = 0
\]

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \\
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial x}
\end{align*}
\]
\[ r = x^2 + xy + y^2 = (x + y)^2 - 2xy = (x + y)^2 + 1 \]

So, we get a triangle:

\[ \phi \quad \text{and} \quad \theta = 90^\circ \]

\[ \sin \theta = \frac{\pi}{2} \frac{\theta}{2} \]

If \( t_1 = t_2 \) then:

\[ 2 \cos \phi = x + y = 1 \quad \text{and} \quad \phi = 45^\circ \]

\[ \sin \phi = \frac{\pi}{2} \frac{\phi}{2} = \frac{\pi}{4} \]

If \( t_1 \neq t_2 \) then:

\[ \phi = \frac{\pi}{2} \]

\[ \sin \phi = 1 \]

The integral is 1 over:

\[ 0^2 + 0 \cdot T \cdot \pi \]

\[ 0 = \frac{\pi}{2} \]

\[ \frac{2}{\pi} \int_0^{\pi/2} \cos \theta \, d\theta = 0 \]
\[ A(+) = \ldots \]

\[ x^4 + x^2 = 0 \quad \Rightarrow \quad x(2 + x^2) = 0 \]

\[ 2x + x^2 = 0 \quad \Rightarrow \quad x(x + 2) = 0 \]

\[ x = 0 \quad \text{or} \quad x = -2 \]

Hence, the solutions are:

\[ x = 0, -2 \]

For the second part:

\[ \frac{dx}{dt} = f(\frac{x}{y}) \cdot \frac{dy}{dt} \]

\[ \frac{dx}{dt} = f(\frac{x}{y}) \]

\[ \frac{dy}{dt} = \frac{x}{y} \]

\[ \frac{d}{dt} \left( \frac{x}{y} \right) = f(\frac{x}{y}) \left( \frac{x}{y} \right) - \frac{x}{y} \cdot \frac{dx}{dt} \]

\[ \frac{d}{dt} \left( \frac{x}{y} \right) = f(\frac{x}{y}) \cdot \frac{x}{y} - \frac{x}{y} \cdot f(\frac{x}{y}) \]

\[ \frac{d}{dt} \left( \frac{x}{y} \right) = 0 \]

Integrating both sides:

\[ \frac{x}{y} = c \]

where \( c \) is a constant.

The equation \( x^4 + x^2 = 0 \) implies:

\[ x(2 + x^2) = 0 \]

\[ x = 0, -2 \]

Thus, the solutions to the equation are:

\[ x = 0, -2 \]

The function \( f(\frac{x}{y}) \) is defined as:

\[ f(\frac{x}{y}) = \frac{x}{y} \]

For the case where \( \frac{x}{y} = 0 \):

\[ \frac{x}{y} = 0 \]

\[ x = 0 \]

For the case where \( \frac{x}{y} = -2 \):

\[ \frac{x}{y} = -2 \]

\[ x = -2y \]

Therefore, the solutions are:

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\[ \frac{x}{y} = -2 \]

\[ x = -2y \]

Therefore, the solutions are:

\[ x = 0, -2 \]
\[ x(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{3t} + C_4 e^{-4t} \]

Each \( C_i \) can be found by...

\[ x(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{3t} + C_4 e^{-4t} \]

\[ 2x(t) + 2x(t) - 2zt = 0 \]

Thus \( x(t) \) is a solution when...

\[ x(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{3t} + C_4 e^{-4t} \]

Now consider ...
\[ x^2 + x > 0 \]

Proof: Let \( x = 0 \) be a solution.

\[ 0 = g(0, 5x) + \left( \frac{11x}{9} \right)(1) = x^2 + 6x + 7 \]

\[ 0 = (x + 6)^2 + 7 \]

\[ x = -6 \pm \sqrt{7} \]

Since \( x \) is a real number, \( x = -6 + \sqrt{7} \) is a valid solution.

Therefore, the solution set is:

\[ x = -6 + \sqrt{7}, -6 - \sqrt{7} \]
NOT AN EXTERMINA
AND IS (OPT COUNTING POLICING)
Thus, if \( f(x) = x^2 \), then \( f'(x) = 2x \).
Now \( f''(x) > 0 \).

Consider \( g(x) = \frac{1}{x} \).

Linear (First Order Approx.)
We can do this case, since \( f''(x) > 0 \).
We have \( g'(x) = -\frac{1}{x^2} \).
Furthermore, \( g''(x) = \frac{2}{x^3} > 0 \).
\[
\begin{align*}
\int_0^x \left( \phi \left( \frac{3\pi}{2} \right) - \phi \left( \frac{3\pi}{2} \right) \right) \, dt &= \phi \left( x \right) - \phi \left( x \right) \\
&= 0
\end{align*}
\]
Consider the case where \( \phi \) is a constant. Then, the integral becomes:

\[
\int_0^x \left( \phi \left( \frac{3\pi}{2} \right) - \phi \left( \frac{3\pi}{2} \right) \right) \, dt = \phi \left( x \right) - \phi \left( x \right) = 0
\]

Let's look at another case where \( \phi \) is not a constant function:

\[
\frac{\partial}{\partial t} \left[ \phi \left( \frac{3\pi}{2} \right) - \phi \left( \frac{3\pi}{2} \right) \right] = 0
\]

This gives the constant function with variation of function of terminal times fixed.
\[
\begin{align*}
\text{Please fill in 1219.} \\
\frac{\frac{x}{2} + \frac{y}{2}}{\frac{x}{2} + \frac{y}{2}} &= \frac{5}{4} \\
\Rightarrow \quad 3x &= 4y \\
\Rightarrow \quad x &= \frac{4}{3}y \\
\text{(1)} \\
\text{Use N.C. Part Integration.}
\end{align*}
\]
\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\end{align*}
\]
\[ x(0) = c = 1 \]

\[ x = x \pm \sqrt{c - x} \]

\[ 2x \left( x - 1 \right) = 0 \]

Now \[ x = 0 \] or \[ x = 1 \]

Local \( \text{c} \) u e r L a g r a n g e.

\[ \frac{\lambda}{x} - 3 = 0 \] 

\[ x = 0 \]

\[ f' = \frac{d}{dx} \left( \frac{4x}{5} \right) \]

By inspection, \( \lambda = 0 \) \( \Rightarrow \) \( \lambda = 0 \)

EXAMPLE

\[ x(0) = 0 \]

\[ x(1) = 1 \]

\[ x = 0 \]

\[ x = 1 \]
\[ \begin{align*} 
\frac{1}{\phi} &+ \frac{1}{\phi^3} = \frac{1}{\phi^2} \text{ (1.16)} \\
(\phi^2 - 1) \cdot x - \frac{3}{\phi^2} - \frac{1}{\phi^3} &+ \frac{3}{\phi^3} - \frac{3}{\phi^2} \\
&= 1, \quad y = \phi^2 - 1 \\
&\Rightarrow \phi^2 - 1 = \frac{\phi^2}{\phi^2} \\
&\Rightarrow \phi^2 = 1 + \sqrt{5} \\
&\Rightarrow \phi^2 = \frac{1 + \sqrt{5}}{2} \\
&\Rightarrow \phi = \frac{1 + \sqrt{5}}{2} \\
&\Rightarrow \phi = 1.618033988749895 \\
&\Rightarrow \phi^2 = 2.618033988749895 \\
\end{align*} \]

We consider minimizing

\[ \text{optimal solution} \]

AN
HOMEWORK

\[ \text{CONDITION} \]

\[ \text{DERIVATIVE} \]

\[ \text{WE RESTRICT} \]

\[ \text{0} = \]

\[ \begin{align*}
\frac{\text{df}}{\text{d}x} &= 0 \\
\frac{\text{d}^2 f}{\text{d}x^2} &= -1 \\
\text{0} &= \frac{\text{df}}{\text{d}x}
\end{align*} \]
\[
\begin{align*}
\frac{d^2x}{dt^2} + \frac{5}{x} & = (2 - 12x^2 - 12x^2) = 0 \\
\Rightarrow x = -2 & = c_1 \\
\Rightarrow x = a & = c_2
\end{align*}
\]

For the Euler-Lagrange equation, it becomes:

\[
\begin{align*}
x(t) = x(b) [1 - \phi(t)] = x(b) \int_0^t \frac{\phi(t) dt}{\phi(t)}
\end{align*}
\]

subject to:

\[
\begin{align*}
x(0) = 0 \\
x(2) = 1
\end{align*}
\]

Find a piecewise smooth curve that minimizes the action.

Weierstrass-Caratheodory condition.
Thus \( x' \) is \( \frac{2x}{x^2 - 1} \).

So \( \frac{2x}{x^2 - 1} \) is the second corner condition.

Any set of these \#s must hold.

Let's look second corner condition.

\[ \begin{align*}
\frac{2}{x^2 - 1} &= x' \quad \text{(1)} \\
\frac{2}{x^2 - 1} &= x' \quad \text{(2)} \\
\frac{2}{x^2 - 1} &= x' \quad \text{(3)} \\
\frac{2}{x^2 - 1} &= x' \quad \text{(4)}
\end{align*} \]
\[\begin{align*}
1.0 & \leq x, \quad 0 < y \\
\frac{3}{2} x - 1 & = 0 \\
0 & = 3 - 2x \\
\sqrt{2} = 2 &
\end{align*}\]
\[ \frac{8}{2} = 2 \cdot \frac{\sqrt{2}}{2} \cdot (\frac{\sqrt{2}}{2}) \]

\[ \sin = \frac{1}{2} \]

\[ \text{solve} = \frac{1}{2} \]

\[ \text{slope} = \frac{1}{2} \]

Optimal Solution: A straight line if we wanna smooth curve, with \( y = c \) and \( c = \frac{1}{2} \). Optimal is obviously \( c = 0 \).

\[ f = \frac{1}{2} \cdot x \cdot (1 - x) \]

Another optimal line.
\[ f(x, y) = \sqrt[3]{x} + y \]

**Multiplier Theory**

Use multipliers to use Lagrange methods. The problem becomes unconstrained plus into it plus 0.5 \[ f(x', y') = 0 \]

\[ x = f(t) \]

\[ y = f(t) \]

**Constraints:** \[ f(w', t) = 0, w = 1, w' \in \mathbb{R} \]

**Condition:**

Recall: Lagrange Multipliers

Indepedent, linearly independent nonlinear functionals:

\[ f(x), f'(x), \ldots \] are regular point if these constraints is said to be a regular point. Test constraints (star) is a point that satisfies constraints \[ f'(x) = 0, x = 1, \ldots \]

Nonlinear constraints:

To optimize subject to nonlinear constraints
SAME EQUATIONS,
$$\Phi_0 = \phi + \int (w, m) dt$$

THEN SIMILARLY
$$+ f = f (w, m', t) = 0$$

Look @ differential constraint

$$\begin{cases}
(6), \quad \frac{\partial}{\partial t} \frac{\partial \phi}{\partial t} \\
\frac{\partial}{\partial t} \frac{\partial \Phi}{\partial t} = 0
\end{cases}$$

Constraint point
$$\begin{bmatrix}
0 = \frac{\partial}{\partial t} (\phi + (m, w) \phi) \\
0 = \frac{\partial}{\partial t} (\phi + (m, w) \phi)
\end{bmatrix}$$

(7)
$$\phi = \phi (w, m', t) + \int f (w, m', t) dt$$

Note: This is what we do.

NECESSARY CONDITIONS:

$$\frac{\partial}{\partial t} \frac{\partial \phi}{\partial t} + f = 0$$

$$\frac{\partial}{\partial t} \frac{\partial (\phi + (m, w) \phi)}{\partial t} = 0$$

$$+ f = f (w, m') + \int \left[ \phi \left( \frac{m, w}{\phi} \right) + \frac{\partial}{\partial t} \frac{\partial \phi}{\partial t} \right] dt$$
EXAMPLE (SIMPLE) ROCKET PROB

\[ \dot{\theta} = u(t) \]
\[ J = \frac{1}{2} \int_0^T (\dot{\theta})^2 \, dt \]
\[ \theta(0) = 1, \quad \theta(T) = 0 \]
\[ \dot{\theta}(0) = 1, \quad \dot{\theta}(T) = 0 \]

\[ x_1 = \theta, \quad x_2 = \dot{\theta} \]

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \theta \end{bmatrix} \]

**NOTE:**

\[ \dot{x}_1 - x_2 = 0 \]
\[ x_2 - u = 0 \]

\[ J_q = \frac{1}{2} \int_0^T [u(t)]^2 + \lambda^T \begin{bmatrix} \dot{x}_1 - x_2 \end{bmatrix} \, dt \]

\[ = \int_0^T \left( \frac{1}{2} (x_2)^2 + \lambda_1 (x_1 - x_2) \right) \, dt \]

\[ \phi_q = \frac{1}{2} u^2 + \lambda_1 (x_1 - x_2) + \lambda_2 (x_2 - u) \]

**LET**

\[ w = \begin{bmatrix} x \\ u \end{bmatrix} \]
The given equations are:

\[ x(t) = \frac{1}{2} t + \frac{1}{2} \]
\[ y(t) = \frac{1}{2} t + \frac{1}{2} \]

They are linear functions of time.

For time \( t = 1 \), we have:

\[ x(1) = \frac{1}{2} \cdot 1 + \frac{1}{2} = 1 \]
\[ y(1) = \frac{1}{2} \cdot 1 + \frac{1}{2} = 1 \]

This gives us the point \((1, 1)\) on the line.

The line passes through the origin when \( t = 0 \), with:

\[ x(0) = \frac{1}{2} \cdot 0 + \frac{1}{2} = \frac{1}{2} \]
\[ y(0) = \frac{1}{2} \cdot 0 + \frac{1}{2} = \frac{1}{2} \]

The optimal solution is when:

\[ x = \frac{1}{2} \]
\[ y = \frac{1}{2} \]

This corresponds to the point \((0.5, 0.5)\) on the line.

Note: The equations and points are consistent with the graph, showing the linearity and the specific points on the line.
Let $f(x, y) = f_{w, t}(x, y)$.

Find $x$ st $f(x, y) = f_{w, t}(x, y) = \min$.

Take $\mathbf{C} = \mathbf{C}$. That is the Lagrange Multipliers Method.

$\sum$ to get $f_{w, t}$.

$\sum$ to get $f_{w, t}$. 

Examine $\min f_{w, t}$ subject to $\mathbf{C}$.
\[ \frac{d}{dt} \left[ \frac{\sqrt{S}}{\sqrt{T}} \right] \]

\[ \sqrt{S} \frac{d}{dt} \left( \frac{S}{T} \right) + \frac{x}{5} \left( \frac{S}{T} \right) \sqrt{T} \]

\[ \int_{0}^{T} \left[ \left( \text{terms}\right) \frac{\sqrt{T}}{\sqrt{S}} \right] \frac{dt}{dt} \]

\[ \left( \text{terms} \right) \frac{\sqrt{T}}{\sqrt{S}} \]

\[ \int_{0}^{T} \left( \text{terms} \right) \frac{dt}{dt} \]

\[ \text{INTEGRATE BY PARTS} \]

\[ \int_{0}^{T} \left[ \left( \text{terms} \right) \frac{dt}{dt} \right] \]

\[ \int_{0}^{T} \left( \text{terms} \right) \frac{dt}{dt} \]

\[ \int_{0}^{T} \left( \text{terms} \right) \frac{dt}{dt} \]

\[ \text{USE LA GRAMIC \& MULTIPLIER} \]

\[ \text{MINIMIZE} \]

\[ \text{Find } x, y, z, t \text{ \& vector} \]

Continue Optimal Control

10.14 - 26 (Fat)
\[ f(x) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \]

\[ g(x) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \]

\[ h(x) = \sqrt{x} + \sqrt{y} + \sqrt{z} \]

\[ \text{Given:} \quad \sqrt{x} + \sqrt{y} + \sqrt{z} = 3 \]

\[ \text{Prove:} \quad \sqrt{x} \sqrt{y} \sqrt{z} = 1 \]

\[ \text{First, note:} \quad \sqrt{x} \sqrt{y} \sqrt{z} = \sqrt{xyz} \]

\[ \text{By AM-GM inequality:} \quad \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{3} \geq \sqrt[3]{\sqrt{xyz}} \]

\[ \sqrt[3]{\sqrt{xyz}} \leq \frac{3}{3} = 1 \]

\[ \text{Thus,} \quad \sqrt{x} \sqrt{y} \sqrt{z} \leq 1 \]

\[ \text{But we know that} \quad \sqrt{x} \sqrt{y} \sqrt{z} = 1 \]

\[ \Rightarrow \quad \sqrt{x} \sqrt{y} \sqrt{z} = 1 \]

\[ \text{Our necessary conditions are:} \]

\[ 0^2 + 0^2 + 0^2 = 0 \]

\[ x = f(x, u) \Rightarrow \text{Autonomous or Time Variant} \]

Explicit Functions of \( t \)

\[ 0 = 1 \Rightarrow \phi \neq f \text{ are not} \]

\[ \begin{align*}
\phi & = \frac{\frac{2u}{3}}{1 + \frac{2u}{3}} \\
\text{cancel } \frac{2u}{3} & = \frac{1}{2} \\
\phi & = \frac{x}{1 + \frac{2u}{3} + \frac{2u}{3} + \frac{2u}{3} + \frac{2u}{3}} \\
& = \frac{\frac{2u}{3}}{1 + \frac{2u}{3} + \frac{2u}{3} + \frac{2u}{3}} \\
& = \frac{\frac{2u}{3}}{4u} \\
& = \frac{1}{2} \\
\phi & = \frac{1}{2} \\
x & = \frac{1}{2} (1 + \sqrt{1 + 4}) \\
x & = \frac{1}{2} (1 + 2) \\
x & = \frac{3}{2} \\
\end{align*} \]

Look at second \( \phi \) solution:

\[ \begin{align*}
\phi & = \frac{\frac{2u}{3}}{1 + \frac{2u}{3}} \\
& = \frac{1}{2} \\
\phi & = \frac{x}{1 + \frac{2u}{3} + \frac{2u}{3} + \frac{2u}{3} + \frac{2u}{3}} \\
& = \frac{\frac{2u}{3}}{1 + \frac{2u}{3} + \frac{2u}{3} + \frac{2u}{3}} \\
& = \frac{\frac{2u}{3}}{4u} \\
& = \frac{1}{2} \\
\phi & = \frac{1}{2} \\
x & = \frac{1}{2} (1 - \sqrt{1 + 4}) \\
x & = \frac{1}{2} (1 - 2) \\
x & = -\frac{1}{2} \\
\end{align*} \]
\[ \begin{align*}
\dot{x} &= f(x, u, t) \\
\lambda &= \phi(x, t) + \int_{t_0}^{t_f} \phi(x, u, t) \, dt \\
m \left[ x(t_0), t_0 \right] &= 0 \\
n \left[ x(t_f), t_f \right] &= 0 \\
5x' \left( \frac{6x}{5x} - \lambda \right) \bigg|_{t_0}^{t_f} &= 0
\end{align*} \]

\[ \lambda = -\frac{54}{5x} \quad \text{Eq. } \quad \lambda = \phi + \lambda' f \]

\[ x = f(x, u, t) = \frac{54}{5x} \lambda \]

\[ \begin{align*}
\frac{54}{5x} &= 0 \\
\lambda(t_0) &= \frac{54}{5x(t_0)} + \left( \frac{6x}{5x} \right)^T V \\
\lambda(t_f) &= \frac{54}{5x(t_f)} + \left( \frac{6x}{5x} \right)^T V
\end{align*} \]

\[ \begin{align*}
x_1 &= x_2 \\
x_2 &= x_3 \\
x_3 &= 0
\end{align*} \]

\[ \text{We wish to derive the system} \]

\[ \text{we reach the terminal} \]

\[ \text{manipulation} \]

\[ x_1(t_1) + x_2(t_2) = 0 \]

\[ \int_0^{t_f} u^2 \, dt \text{ is minimized} \]
We are given

\[ 0 = (x - 1)^2 + (y - 1)^2 \]

so

\[ x = 0 \]

Therefore, the station cost equation becomes

\[ y - 1 = \frac{\theta - 1}{2} \]

from the equation

\[ y = \frac{\theta - 1}{2} + 1 \]

Now, solve for the constants

\[ x^2 + y^2 = \frac{\theta^2}{4} \]

First, find \( \phi \)
Consider the problem:

\[ u(t) = \psi(t) + \int_{0}^{t} f(t,v) \, dv \]

\[ \dot{x}(t) = f(x(t), v(t)) + \int_{0}^{t} (x(t_0) + \int_{0}^{t_0} x(t_0) \, dt_0) \, dt_0 \]

\[ \dot{x}(t) = \phi(t) + \int_{0}^{t} \psi(t) \, dt \]

\[ x(t) = \phi(t) + \int_{0}^{t} \psi(t) \, dt \]

\[ x(t) \text{ free} \]

\[ x(t) = \psi(t) + \int_{0}^{t} f(t,v) \, dv \]

\[ \dot{x}(t) = f(x(t), v(t)) + \int_{0}^{t} (x(t_0) + \int_{0}^{t_0} x(t_0) \, dt_0) \, dt_0 \]

\[ \dot{x}(t) = \phi(t) + \int_{0}^{t} \psi(t) \, dt \]

\[ x(t) = \phi(t) + \int_{0}^{t} \psi(t) \, dt \]

\[ x(t) \text{ free} \]

\[ x(t) = \psi(t) + \int_{0}^{t} f(t,v) \, dv \]

\[ \dot{x}(t) = f(x(t), v(t)) + \int_{0}^{t} (x(t_0) + \int_{0}^{t_0} x(t_0) \, dt_0) \, dt_0 \]

\[ \dot{x}(t) = \phi(t) + \int_{0}^{t} \psi(t) \, dt \]

\[ x(t) = \phi(t) + \int_{0}^{t} \psi(t) \, dt \]

\[ x(t) \text{ free} \]
\[ 0 = \pm \sqrt{\left( \frac{x}{3} \right)^2 + \left( \frac{y}{6} \right)^2 + \left( \frac{z}{9} \right)^2} + \frac{\sqrt{5}}{6} \left( x + \frac{5}{3} + \frac{y}{6} + \frac{z}{9} \right) \]
\[
\frac{x^3}{\sqrt{5}} = c
\]

\[
\left(\frac{x}{c}\right)^3 = x
\]

\[
\begin{align*}
\frac{x}{c} &= f(x/\sqrt{5}) \\
\Rightarrow \quad x &= f\left(\frac{x}{\sqrt{5}}\right)
\end{align*}
\]

\[
\begin{align*}
\gamma &= 0 = 0 + \frac{2\pi}{\sqrt{5}} + \frac{4\pi}{\sqrt{5}} \\
\Rightarrow \quad t &= t = 0 \\
\Rightarrow \quad X &= \frac{x}{c} = \frac{2\pi}{\sqrt{5}} + \frac{4\pi}{\sqrt{5}}
\end{align*}
\]

\[
\begin{align*}
\gamma &= 0 = 0 + \frac{2\pi}{\sqrt{5}} + \frac{4\pi}{\sqrt{5}} \\
\Rightarrow \quad t &= t = 0 \\
\Rightarrow \quad X &= \frac{x}{c} = \frac{2\pi}{\sqrt{5}} + \frac{4\pi}{\sqrt{5}}
\end{align*}
\]

Conditions from this arc.
Our equations to date are

\[ A = \begin{bmatrix} 4 & 5 \\ -2 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A + U = \begin{bmatrix} 6 & 7 \\ 2 & 5 \end{bmatrix} \]

Now, hence \( X = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \)

The linear regulation equation

\[ x = A + BU, \quad y = Cx + Du \]
In order to solve the Riccati equation, we solve a corresponding system of equations.

\[
\begin{align*}
    & (x + \gamma) = 0 \\
    & (x + \gamma) = 0 \\
    & \text{This implies}\ 
    & (x + \gamma) = 0 \\
    & \text{This implies}\ 
    & (x + \gamma) = 0 \\
\end{align*}
\]

Note: the known boundary conditions.

Thus, we get the following system of equations:

\[
\begin{align*}
    & K_p + A_T p + p R - \lambda p - q = 0 \\
    & D_p = -p A_T + p R - \lambda p - q = 0 \\
\end{align*}
\]

Let \( p \) is our unknown.

Now, we will get the Kalman gain

\[
\begin{align*}
    & x(t) = p(t) x(t) + \gamma p(t) x(t) \\
    & s(t) = p(t) x(t) \\
    & x(t) = x(t) \\
\end{align*}
\]

Assume: there exist a correspondence.
\[ \mathbf{P}(t) = \mathbf{P}(t-1) \mu + \mathbf{Q}(t) \]

\[ \mathbf{K}(t) = \mathbf{P}(t) \mathbf{H}^T (\mathbf{H} \mathbf{P}(t) \mathbf{H}^T + \mathbf{R})^{-1} \]

Thus, we may write

\[ \mathbf{P}(t) = \mathbf{P}(t-1) - \mathbf{P}(t-1) \mathbf{K}(t) \mathbf{P}(t) \mathbf{K}(t)^T \mathbf{P}(t-1) \]

\[ \mathbf{P}(t) \mathbf{P}(t-1) = 0 \Rightarrow \mathbf{P}(t) = \mathbf{P}(t-1) \]

\[ \mathbf{P}(t) + \mathbf{P}(t-1) = \mathbf{P}(t) \]

\[ \mathbf{P}(t) \mathbf{P}(t) = 0 \Rightarrow \mathbf{P}(t) = \mathbf{P}(t-1) \]

If \( S = \) is very large,

On Ricatti equation

\[ \text{can determine Kalman Gain, } \mathbf{K}(t) \]

\[ \text{we set } \mu = \text{solve } \text{Ricatti}, \text{so we} \]

\[ \mathbf{K}(t) \]

\[ \mathbf{A} \]

\[ \mathbf{H} \]

\[ \mathbf{P}(\mathbf{t}) \]

\[ \mathbf{Q} \]

\[ \mathbf{R} \]
The original text is legible but contains mathematical expressions and equations. It seems to be a page from a textbook or a notebook, discussing mathematical problems and solutions. Here is a transcription of the text:

**First Order Linear Equation**

To reduce the equation \( y^2 + \frac{dy}{dx} = 0 \) to a variable \( y = y + \frac{1}{x} \)

Let \( y(x) \) be a particular solution.

\[ \frac{dy}{dx} + 2x(x^2 + 4) + 4(x^2 + 4) = 0 \]

**Methods of Solution (1st Order, 1st Degree)**

Consider the function \( f(x) = \frac{dy}{dx} + 2x(x^2 + 4) + 4(x^2 + 4) \)

From \( \frac{dy}{dx} = 0.5t + 1.5 \) constant, \( \frac{dy}{dx} = \frac{1}{2} \).

Solution obtained at earlier, \( \frac{dy}{dx} = 0 \).

Now, \( \frac{dy}{dx} = p + \frac{1}{2} \)

\[ p = \frac{1}{2} + \frac{1}{2} \left( \frac{1}{x} \right) \]

A particular equation in general:

For this problem, we have

\[ y = \frac{\sqrt{2}x + c(t)}{\sqrt{2}x + c(t)} \]

Problem solved.
Evaluate \( k \) by \( P(t + 4) = 5 \):

\[
P = 1 + \ell = 1 + k e^{k t} = 13\]

\[
\ell = k e^{kt} - \frac{5}{2}
\]

General solution is:

\[
\ell = \frac{z}{2} + 3z + 1 - z^2
\]

\[
\Rightarrow -\frac{z}{2} = z + 2z + 1 - z^2
\]

\[
\Rightarrow -\frac{z}{2} = z + 2z + 1 - z^2
\]

Plug in above:

\[
P = \frac{z}{2}
\]

\[
P = 1 + \frac{1}{z}
\]

Let \( p \), then

\[
p = p + p - 2 \Rightarrow p = 2
\]
\[
\begin{align*}
0 &\leq y(t) \\
\text{when } &\ x(t) = 0 \quad \text{only if } \quad x(t) = 0
\end{align*}
\]

\[
T \quad 0 \leq (t') \quad \text{when } \quad y(0) = 0
\]

\[
X' + u = 0
\]

\[
H' = 0
\]

Step Function:

\[
H(s(x', t)) = \text{modifed Heaviside function}
\]

\[
\begin{align*}
&\text{let } \ x'_0 = \text{new equation defined by } y(x') \geq 0 \\
&\text{for } x \geq (x'_0, t) \quad \text{for all } \ t
\end{align*}
\]

**Constraints**

**Control Variable Inequality**

The maximum principle with state
\[ P \begin{bmatrix} 0 & \Phi(t) & \frac{\Phi}{2} \\ 0 & \frac{\Phi}{2} & \Phi(t) \end{bmatrix} \sqrt{\frac{c}{2}} + \\
\begin{bmatrix} (\Phi(t) \times \Phi(t)) & \Phi(t) \times \Phi(t) \\ \Phi(t) \times \Phi(t) & \Phi(t) \times \Phi(t) \end{bmatrix} \Phi(t) = \begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} \]

**Let's solve:**

\[
\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} X \begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} = \begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix}
\]

**Matrix Form:**

\[
\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} = -\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} A^T + \begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix}
\]

Thus:

\[
\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} = (A^T - \Gamma) X = 0
\]

Thus:

\[
\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} = \begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix}
\]

**Thus, if: R = 0, we have a regular process.**

\[
\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \Phi(t) \sin \theta - \Phi(t) \cos \theta \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Phi(t) \sin \theta + \Phi(t) \cos \theta \end{bmatrix}
\]

\[
\begin{bmatrix} \Phi(t) \\ \Phi(t) \end{bmatrix} = \min(t) \begin{bmatrix} \Phi(t) \sin \theta + \Phi(t) \cos \theta \end{bmatrix}
\]

**Tracking Problem:**

\[
x = \min(t) \begin{bmatrix} \Phi(t) \sin \theta + \Phi(t) \cos \theta \end{bmatrix}
\]

10-25-76 (Mon)
\[ x(t) = \phi(t) x(t_0) + \int_{t_0}^{t} \phi(t,s) f(s) \, ds \]
\[ u = K^T x(t) \]

\[ = p x(t) + p_0 \]

\[ + \sum_{i=1}^{n} \left[ p_i \Phi_i x(t) \right] \]

\[ = p(x - r) + \sum_{i=1}^{n} \Phi_i x(t) \]

\[ = p(x - r) + \Phi x(t) \]

\[ = (x - r)^T \Phi^T u \]

\[ = (x - r)^T \Phi^T (K^T x(t)) \]

\[ = (x - r)^T K \Phi x(t) \]

\[ = \sum_{i=1}^{n} (x - r)^T \Phi_i x(t) \]

\[ = \sum_{i=1}^{n} (x_i - r_i)^2 \]

\[ \Rightarrow u \text{ is our optimal control} \]

Once we set \( u = K^T x(t) \), we have the expression for the optimal control in the form of a quadratic function.
\[ \mathbb{P}(T \leq T^*) = 0 \]

**Penalty Term:**

PENALTY TERM: 

**PENALTIES TO RULE IN:**

PROBLEM IS ESSENTIAL SOLVING.

\[ \frac{d}{dt} \left( \frac{y}{a} \right) = -5 \tau(t) \]

\[ P(f) = 5 \]

\[ \frac{1}{2} \left( \text{AT}^3 + \text{PBR} \right) + 3 - 0 \]

\[ \text{SAME AS REAL PROBLEM} \]

\[ P+\text{PDA} + \text{AT}^3 - \text{PBR} = 0 \]

**In General, must n=0?**
THE BOLZA PROBLEM
(CONTROL & STATE INEQUALITY
CONSTRAINTS)

\[ J = G(x(t_f), t_f) + \int_{t_0}^{t_f} \phi(x, u, t) \, dt \]

SUBJECT TO

i. \[ x = f(x(t)) \]

ii. \[ N = (x(t_f), t_f) \rightarrow 0 \] END POINT

iii. \[ M(x(t), t) = 0 \] INITIAL COND

iv. \[ x(0) = [x(0)] \]

v. \[ g(x, u, t) \geq 0 \] CONTROL INEQ

vi. \[ r(x(t), t) \geq 0 \] STATE INEQ. CON.
\[
I = \int_{-\infty}^{\infty} \sqrt{(\frac{\partial x}{\partial \theta})^2 + (\frac{\partial x}{\partial \phi})^2} d\phi
\]

\[
\frac{\partial}{\partial \theta} \left[ \frac{1}{2} \varepsilon \left( (\frac{\partial x}{\partial \theta})^2 + (\frac{\partial x}{\partial \phi})^2 \right) \right] = 0
\]

\[
\frac{\partial}{\partial \phi} \left[ \frac{1}{2} \varepsilon \left( (\frac{\partial x}{\partial \theta})^2 + (\frac{\partial x}{\partial \phi})^2 \right) \right] = 0
\]
Again

\[
(\ell_1^2 - \ell_2^2 - \ell_3^2 - \ell_4^2) = 0
\]

\[
\ell_1^2 = \ell_2^2 + \ell_3^2 + \ell_4^2
\]

Now, as we've shown, \( \ell = \mathbf{O} \).
Thus, making these substitutions, we

\[ \phi(t) = \frac{1}{2} t \]

Conside\n
\[ c = \left( \frac{S}{2 T} \right) \]

Consider\n
\[ 0 \leq t \leq T \]

Conside\n
\[ 0 \leq t \leq T \]
\[
\frac{x}{\phi^2} = \frac{x}{\phi^3} + \phi
\]

\[
\phi + x = \frac{3}{y}
\]

We wish to solve for \( P \)

\[
0 = \left( \frac{\phi x}{\phi^3} + \frac{x}{\phi^2} \right)
\]

\[
0 = \frac{\phi x}{\phi^3} + \frac{x}{\phi^2}
\]

\[
\phi = x \Rightarrow 0 = \frac{x}{\phi^3} + \phi
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]

\[
\frac{\phi x}{\phi^3} + \frac{x}{\phi^2} = 0
\]

\[
\frac{\phi x}{\phi^3} + \frac{x}{\phi^2} = 0
\]

\[
\frac{\phi x}{\phi^3} + \frac{x}{\phi^2} = 0
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]

\[
\phi = x \Rightarrow 0 = \frac{x}{\phi^3} + \phi
\]

\[
\phi = x \Rightarrow 0 = \frac{x}{\phi^3} + \phi
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]

\[
\frac{x}{\phi^3} + \phi = 0
\]
\[ x = 10 \sin (x \cdot t) \]

\[ y = \cos (x \cdot t) \]

Recall common conditions
OPTIMAL CONTROL

This is a necessary condition for optimal control.

\[ f'(x, t) \geq f(x, t) \]

Thus:

\[ \frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \geq 0 \]

\[ x^2 + y^2 - (z - \frac{t}{2})^2 \geq 0 \]

Now, \( x \) is an optimal control vector.

\[ \frac{\partial f}{\partial x} \]

\[ f = \frac{1}{2} x^2 + \frac{1}{2} y^2 - \frac{1}{2} z^2 - \frac{1}{2} t^2 \]

\[ \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \]

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Example 5: Optimal Control

\[ \begin{align*}
0 < x \leq y \leq c
\end{align*} \]

Then \[ \begin{align*}
x^2 + C(0)^2 & \geq C(x^2 + \beta_1) \\
& \geq \frac{1}{2} \left( \frac{x}{\gamma_1} \right) \frac{1}{2} \left( \frac{x}{\gamma_2} \right)
\end{align*} \]

\[ \begin{align*}
x^2 + y^2 & = 0 + \sqrt{\frac{x}{\gamma_1} \frac{x}{\gamma_2}}
\end{align*} \]

Consider \[ \begin{align*}
\begin{cases}
\begin{align*}
0 & \leq (x', y') \leq (c', c') \\
\alpha & = 0
\end{align*}
\end{cases}
\end{align*} \]

Use P.S. Max Principle:

\[ \begin{align*}
\text{Find } u = \text{Optimal Control}
\end{align*} \]

With \[ \begin{align*}
\text{Subiect to: } v = F \quad \text{ minimized}
\end{align*} \]

\[ \begin{align*}
\text{Example: } u = \gamma_1
\end{align*} \]

\[ \begin{align*}
2 \gamma_1 = 1
\end{align*} \]

\[ \begin{align*}
0 = \gamma_1 \left( \frac{x}{\gamma_1} \right) \left( \frac{x}{\gamma_2} \right)
\end{align*} \]

\[ \begin{align*}
N(x_0, x_0) = x(t) = 0
\end{align*} \]

\[ \begin{align*}
(0, 0) = 0
\end{align*} \]

\[ \begin{align*}
\text{Now, } \gamma_1 = 1
\end{align*} \]

\[ \begin{align*}
0 = \gamma_1 + 1 \left( \frac{\gamma_1}{\gamma_2} \gamma_2 \right)
\end{align*} \]

\[ \begin{align*}
\text{Also, let } x(t) = 0
\end{align*} \]

\[ \begin{align*}
\text{we will show } \dot{x} + \dot{u} + \frac{1}{2} \left( \frac{x}{\gamma_1} \frac{x}{\gamma_2} \right)
\end{align*} \]

\[ \begin{align*}
\dot{x} + \dot{u} + \frac{1}{2} \left( \frac{x}{\gamma_1} \frac{x}{\gamma_2} \right)
\end{align*} \]

\[ \begin{align*}
\dot{x} + \dot{u} + \frac{1}{2} \left( \frac{x}{\gamma_1} \frac{x}{\gamma_2} \right)
\end{align*} \]
Thus, we wanna minimize.

\[
\begin{align*}
&v_1 = b_1 \\
&v_2 = b_2 \\
&v_3 = b_3 \\
\end{align*}
\]

\[
\begin{align*}
&u_1 = v_1 \\
&u_2 = v_2 \\
&u_3 = v_3 \\
\end{align*}
\]

We required that

\[
\|u(t)\| \leq 1
\]
(For $h = 1$, we get Newton's method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

We use the method for $y$.

$$f = \frac{y}{x}$$

$$f' = \frac{x - y}{x^2}$$

$$0 \leq x \leq 2$$

$$f(0) = \frac{0}{2} = 0$$

$$f(x) + \frac{x}{2} = x$$

$$\frac{2}{x} \geq 1$$

$$\frac{1}{x} \geq 2$$

Minimum is $x = 1$.

**Example:**

$$y = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$x = -2$$

$$y = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

Optimal control is $u = 0$.
Continuously differentiable function.  

$f(c) = 0$  

$\int x(c(t)) \, dt$  

$x(c) = f$  

$\int f(x(t)) \, dt$  

We want $f(x(t)) = 0$  

$f(t) = -f + u$  

Back to Problem
Solution is

Homo: \( \int (z-t)^2 dz = \int -z^2 + \text{particular} \)

\[ \phi(t) = x - t \]

\[ \dot{v} = \ddot{v} = 0 \]

\[ \alpha = 0, \quad \beta = 0 \]

**Contact Equation**

Let \( j = \frac{3}{2} \)

**Parameter** A more specific

\[ \phi(0) = 0 \]

\[ \phi(t) = \begin{cases} 0 & t < 0 \\ \frac{3}{2} & t \geq 0 \end{cases} \]

Let \( j = 0 \)

\[ \phi(t) = \begin{cases} 0 & t < 0 \\ \frac{3}{2} & t \geq 0 \end{cases} \]

\[ (x' + x) [x(t)] = 0 \]

\[ f(x(t)) = 0 \]

**Conclusion** A
\[ K = 0, 1, 2, \ldots, N - 1 \]

Using Euler's formula, we will be our solution:

\[
x' = \frac{x - y \cos (c - t)}{2} + \frac{x + y \sin (c - t)}{2} + \frac{x}{2} - \frac{y}{2} \frac{1}{\sqrt{(2 - t)^3}}
\]

Integrate.

Sign change should be accounted for.

And so, to no connection, we must integrate this:

\[
x = \frac{1}{2} x (t) - \frac{1}{2} \left( \frac{x}{2} \right) (t - 1) + \frac{1}{2} \left( \frac{x}{2} \right) (t + 1) - \frac{1}{2} \left( \frac{x}{2} \right) (t - 2)
\]

Now solve for \( x(t) \).

Note: Through we can't here.

Then find out the optimal trajectory is...
REFERENCES

Step Six
So we have a time variable
\[ x_{k+1} = x_k - \frac{1}{n} \left( x_k - f(x_k) \right) \]
Compare to Euler's
\[ x_{k+1} = x_k - \frac{h}{2} \left( f(x_k) + f(x_{k+1}) \right) \]
Now

\[ x_0 \leq \frac{2}{C} \left( 1 + \frac{L}{2} \right) \]

Assume \( T_f = 1 \text{ sec} \) \( \Rightarrow N = 25 \)

Solutions

\[ f(x) = x^2 - 6 = 0 \]
\[ f(x) = x^2 - x^2 = 0 \]
EXAM BLE
\[
0 = 1 + x \\
0 = \frac{1 + x}{2} - \frac{1}{2} \\
0 = \frac{x}{2} - \frac{1}{2} + x \frac{1}{2} \\
0 = \frac{x}{2} + \frac{1}{2} \\
0 = \frac{x}{2} - \frac{1}{2} \\
0 + \frac{1}{2} = \frac{x}{2} \\
\frac{1}{2} = \frac{x}{2} \\
1 = x \\
\]

So we get
\[
0 = 1 + x \\
0 = x \\
\]
\[ 0 = \left( \frac{mS}{8S} \right) - \frac{m^2}{8S} \] (2)

\[ 0 = \begin{bmatrix} \frac{x_2}{2} \\ \frac{x_2}{2} \\ \frac{x_2}{2} \end{bmatrix} + \frac{m^2}{8S} \] (3)

\[ t^2 - 1 = \frac{x_2}{2} + \frac{x_2}{2} + \frac{m^2}{8S} = y(5) \]

\[ 0 = \frac{m^2}{8S} \] (4)

From previous problem

\[ 0 = \frac{m^2}{8S} = \frac{m^2}{8S} \] (5)
The result is based on calculating the derivative of the function $f(x, t)$ with respect to $x$.

Thus, the normal forms:

$$
\frac{\partial}{\partial x} (f(x, t)) = 0, \quad \forall \in \text{domain}
$$

$$
\forall x, t \in \text{domain}, \quad f(x, t) = 0
$$

Now, let $g(x, t) = 0$.

$$
0 \leq g(x, t) < 0
$$

$$
0 \leq u \leq b
$$

We must bound controls $u$ such that:

$$
0 \leq \text{controls} \leq b
$$

Or, can control $u$ such that:

$$
\Rightarrow u = u_0 + v_1 + G(t)
$$

$$
\Rightarrow f(x, t) + h(t, x) + g(t, x) = 0
$$

$$
\Rightarrow H(x, v) = h(t, x) + g(t, x)
$$

$$
\Rightarrow H(x, v(t)) = h(t, x) + g(t, x)
$$

Control:

If $u$ will be minimum control:

$$
\Rightarrow u = u_0 + v_1 + G(t)
$$

Hamiltonian -

$$
H(x, v(t)) = \Theta_0 (x(t) + y(t) + \theta(t))
$$

$$
\Rightarrow H(x, v(t)) = f(x(t), v(t)) + c(x, t) + \theta(t)
$$

Bang - Bang Control
Closed Book

Test will cover up to 

\[ \phi = \frac{\varphi}{\chi^b} \]

\[ x_{1b} \leq x_{1b} \]

\[ x_{1} + x_{2b} \leq 1 + x_{1b} \]

Using Max Principle

\[ x_{1} = 1 + \sqrt{x_{1} + b_{u}} \]

\[ \exists \quad I \subseteq C \]

\[ J = \sqrt{t} \]

Minimum Time

\[ x = v_{x} + b_{v} \quad v_{x} \text{ is a scalar} \]

Minimum Time Problem

\[ x = \frac{\dot{c} - \dot{c}_{0}}{\ddot{c}} = \frac{\dot{s}}{\ddot{s}} \]

In order to solve the
\[ x = A \xi + b \xi \]

**Minimum Time Problem**

\[ H(x, u(t)) \leq H(x, u(t)) \leq t_{f} \]

**Substitute into our ODE:**

\[ x \xrightarrow{t} x(t) = x_{0} \quad x(0) = x_{0} \]

\[ x_{f} = A \xi + b \xi \]

**Solution:**

\[ x = A \xi + b \xi \]

\[ H(x, u(t)) = H(x, u(t)) = t_{f} \]

\[ x = t_{f} (A \xi + b \xi) = 1 \]

\[ \Rightarrow \phi = 1 \]
\[
\begin{align*}
\text{Step 1:} & \quad \int_{t}^{t+T} b \, dt \quad \text{and} \quad \int_{t}^{t+T} e^{r} \, dt \\
\text{Step 2:} & \quad \text{Evaluate the integrals.} \\
\text{Step 3:} & \quad \text{Solve for} \quad x(t) \quad \text{and} \quad \phi(t). \\
\end{align*}
\]
\[
\frac{\frac{1}{x+y} + \frac{1}{x+z}}{\frac{1}{y+z}} = G
\]

When \( S \) and \( S^2 \) are both positive, then \( G \) is positive.

\[
S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Now consider \( C \) is positive.

\[
C \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
x = (0,x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}
\]

\[\text{Example}\]
\[
\begin{align*}
&0 < s - t = 0 \\
&b_p = b_{(1,2)} - 1 \\
&\frac{b}{1} = \sqrt{\frac{b}{1}}
\end{align*}
\]
\[ 0 = 1 - 2x + 2x^2 + x^3 \]

\[ \begin{align*}
 0 & > 1 - 2x + 2x^2 + x^3 \\
 0 & > \frac{1}{x^2} \left[ \frac{1}{2} \right] \\
 0 & > \left[ \frac{1}{x^2} \right]
\end{align*} \]

Consider (a):

\[ \begin{align*}
 0 & > \frac{1}{x^2} \left[ \frac{1}{2} \right] \\
 0 & > \left[ \frac{1}{2x^2} \right] \\
 0 & > \left[ \frac{x^2}{2} \right]
\end{align*} \]

\[ \begin{align*}
 0 & > \left[ \frac{x^2}{2} \right] \\
 0 & > \left[ \frac{x^2}{2} \right] \\
 0 & > \left[ \frac{x^2}{2} \right]
\end{align*} \]

\[ \begin{align*}
 0 & > \left[ \frac{x^2}{2} \right] \\
 0 & > \left[ \frac{x^2}{2} \right] \\
 0 & > \left[ \frac{x^2}{2} \right]
\end{align*} \]
\[ x^2 = 0 \leq x < 1 \]  
\[ x = 1 \]

Now:
\[ x = 1 \]
\[ \Rightarrow x^2 + x = 1 \]

Use point's max principle:
\[ x = 0 \]
\[ x = \frac{t}{2} \]

\[ e^t \]

\[ x(1) = 1 \]

\[ x(1) = -1 \]

\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]
\[ x^2 + y^2 = 1 \]

\[ x(t) = \frac{1}{2} x^2 - \frac{1}{2} y^2 + C \]

\[ y(t) = \frac{1}{2} x^2 + \frac{1}{2} y^2 + C \]

\[ x_0(t) = \pm t + C \]

\[ x_0 = x_2 = \frac{t}{t^2 + z^2 - t^2 + C \pm C} \]
If we are on A or B, we can set $\Delta y = 0$ to ZERO, center $A_0$.

Now $x = -1$, $\beta = 0$. $x(0) = -1$.

Comes to $C_\Delta = \frac{1}{2} x^2 - \frac{1}{2} x^2$.

$x(t) = \frac{2}{3} x^2$.

$x'(x) = \frac{2}{3} x^2$.

$x = -1$, $c = 1$.

If we are on A or B, we can set $\Delta y = 0$ to ZERO, center $A_0$. 

Now $x = -1$, $\beta = 0$. $x(0) = -1$.
\[ F = k + M \cdot \sin \theta + g \cdot \cos \theta \]

\[ \theta = \frac{v^2}{2g} \cdot \sin \theta + \frac{v^2}{2g} \cdot \cos \theta \]

\[ m = M \cdot \cos \theta - M \cdot \sin \theta \]

\[ \Rightarrow m = M \cdot \cos \theta - M \cdot \sin \theta \]

\[ \Rightarrow m = M \cdot \cos \theta + M \cdot \sin \theta \]

\[ \Rightarrow m = M \cdot \cos \theta - M \cdot \sin \theta \]

\[ \Rightarrow m = M \cdot \cos \theta + M \cdot \sin \theta \]

\[ \Rightarrow m = M \cdot \cos \theta - M \cdot \sin \theta \]

\[ \Rightarrow m = M \cdot \cos \theta + M \cdot \sin \theta \]
\[ M = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \]
\begin{align*}
L = [1, x_1, x_1^2] \\
T = [1, x_2, x_2^2] \\
\lambda = [0, x_1, x_1^2 + 1] \\
\lambda = [0, x_2, x_2^2 + 1]
\end{align*}

\text{Example}

\begin{align*}
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\end{align*}

\text{Last time we plotted } S = 6 = 1.

\text{For } A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}

\text{column space } S = \mathbb{R}^2.
We worked backwards.

This is some problem.

\[ x' \cdot 1 = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c \\
\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ x = 1, \quad y = 7 \]

\( \Rightarrow \) for \( v = 1 \)

\[ D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

\[ X + 2X \cdot \frac{3}{2} - \frac{3}{2}X \]

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \begin{cases} x' \cdot 1 = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases} \]

\[ x = 1, \quad y = 7 \]

\[ \Rightarrow \] for \( v = 1 \)

\[ D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

\[ X + 2X \cdot \frac{3}{2} - \frac{3}{2}X \]

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \begin{cases} x' \cdot 1 = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases} \]

\[ x = 1, \quad y = 7 \]

\[ \Rightarrow \] for \( v = 1 \)

\[ D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

\[ X + 2X \cdot \frac{3}{2} - \frac{3}{2}X \]

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\begin{align*}
\text{Consider the case for } \alpha = 0. \quad & \text{solve for } \lambda. \\
\end{align*}
\[ a + b = c - d \]

\[ a = 0 \quad x = 0 \]

\[ \begin{pmatrix} \frac{a^2}{2} & \frac{a}{2} \\ \frac{a}{2} & \frac{1}{2} \end{pmatrix} \]

\[ \frac{1}{2} \ln (x + b) = \frac{x^2}{2} \]

\[ y = x^2 - t \]

Pause to solve for \( y \) \[ x(t_2) = x(t_1) + x(t_2) \]

\[ x(t_1) + x(t_2) + x(t_1) + x(t_2) \]

Consider:

\[ x(t_1) = x(t_2) + x(t_1) \]

\[ x(t_1) = x(t_2) + x(t_1) \]

\[ x(t_1) = x(t_2) \]

Last we have, we showed that:

\[ x(t_2) = \frac{t}{2} X(t_1) \]

At \( t = t_2 \) we wanna find \( t - t_2 \)

Consider, the \( \omega = 0 \) \( \theta = 0 \)
\[ y = x' + \frac{1}{t+1} (1 + 1) \]

Therefore, \( Y = x' + 1 \).

At \( t = 0 \), \( x = 0 \), \( \mu = 0 \), \( x' = 0 \), \( A x = 0 \), \( B = 0 \).

\( A = 1 \) and \( T = 1 \), so \( \lambda = 0 \).

Thus, \( x = 1 \).
\[ x = \frac{1}{c} - \frac{c}{t} + C \]

Assume \( t = 0 \) and \( x = 0 \):

\[ x = \frac{1}{c} \]

\[ 0 = \frac{1}{c} - c \]

\[ \frac{1}{c} = c \]

\[ c = \pm 1 \]

When \( A = 0 \), \( x = x(0) = 0 \):

\[ \frac{1}{c} = 0 \]

\[ c = 0 \]

\[ x = 0 \]

Find \( u \) to obtain \( x(t, 0) = 0 \):

\[ u = -\frac{1}{t} \]

\[ x = \frac{1}{c} \]

Example

 indefinite

\[ \int \frac{1}{t} \, dt = \ln |t| + C \]

\[ A \neq 0 \]

\[ x = \frac{1}{c} \]

\[ \frac{1}{c} = c \]

\[ c = 0 \]

\[ x = 0 \]

\[ t = 0 \]

The singular problem

\[ \int \frac{1}{t} \, dt = \ln |t| + C \]
\[
X = x^2 + x + 2 \quad (x \geq 0)
\]

\[
(0 \leq y = 2 + x + 2x + x^2 = \frac{2 + x^2}{y} + x + \frac{x}{4} + 2y = \frac{x^2}{y} + x + \frac{x}{4} + 2y
\]

\[
\begin{bmatrix}
0 & -2x & 0 \\
-1 & x & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

\[
(\forall x \in (1, 4), x = x_0, (x_0) = x_0, (x_0) = x_0, (x_0) = x_0, (x_0) = x_0)
\]

\[
\begin{bmatrix}
1 & -2x & 0 \\
0 & x & 0 \\
-1 & 0 & 0
\end{bmatrix}
\]

\[
1 = \frac{1}{2} \sqrt{2} \quad \text{CONSIDER}
\]

\[
X = \frac{1}{2} \quad \text{OPTIMAL}
\]

Thus, for optimal control,

\[
(\text{OPT})
\]

\[
\text{HEN}
\]

\[
\text{SINGULAR PROBLEM}
\]
Solve we also gotta

\[
\frac{\partial^2 y}{\partial x^2} = x^2
\]

Thus 2nd order ODE.

On the singular arc:

The N critical points (closer loop)

\[ x = x^2 + y = 0 \]

Using straight cuts:

\[ x' = \frac{-x}{y} \]

From (\[ x = \frac{1}{y} \]

\[ x' = \frac{y}{x} \]

\[ \frac{4}{4} \]

\[ x = y^2 \]

\[ x = y^2 \]

Singular solution when
\[ x^2 + 2x + 1 = \frac{1}{2} x - y = C = \text{const.} \]

\[ z = \frac{1}{2} x - y, x^2 = C = \text{const.} \]

\[ \text{On the singular arc} \quad AC = \]

\[ \text{For a NOT explicitly a function, of time, \( t = C \).} \]

\[ \text{For} \quad A = \frac{1}{2}, \quad \frac{1}{2} = \frac{1}{2} \]

\[ x^2 + 2x + 1 = \frac{1}{2} x - y = C = \text{const.} \]

\[ \text{At} \quad x(t) = \frac{1}{2}, \quad \frac{1}{2} = \frac{1}{2} \]

\[ \text{Singular solution exists.} \]

\[ \text{Singular problem when} \]

\[ x = \frac{1}{2} x - y = C = \text{const.} \]

\[ x(t) = \frac{1}{2} x - y = C = \text{const.} \]

\[ t = \tan \theta, \quad x^2 = C = \text{const.} \]

\[ t = \tan \theta, \quad x^2 = C = \text{const.} \]
\[
x^2 = x(t) \frac{d}{dt} (t - t')
\]

**Solve:***

\[
x^2 = \left[ x \right]^2 = \left( t - t' \right)^2
\]

**Equation:**

\[
x^2 = \left( t - t' \right)^2
\]
Thus, we have \( k = x_2 \).

Solve for \( x_1 \):

\[
x_1 = k \pm \sqrt{k^2 - 1}
\]

For \( k = 1 \):

\[
x_1 = 1 \pm 1 = 2, 0
\]

Consider case when \( k < 1 \).
Show in above graph.

More H/P conditions still considered.

Can be negative.

If $z=1$:

\[
\begin{align*}
x(t) &= (x_0 - x_T) + x_T \\
y(t) &= (y_0 - y_T) + y_T \\
z(t) &= (z_0 - z_T) + z_T \\
x_1 &= \frac{1}{x_0} \sqrt{x_0^2 - x_T^2} \left( \frac{x_T}{x_0} + x_T \right) \\
\end{align*}
\]
\begin{align*}
0 &= x^2 + x + 1 \\
0 &= x(x + 1) \\
0 &= x + 1 \\
T &= \infty
\end{align*}

\text{We have fixed } T. \text{ Let } T \to 0.
The solution to this equation is balanced.

Assume \(-1 < u < 1\) and consider

\[ \int_{-1}^{1} x^2 + 1 \, dx = 0 \]
\[ x^2 - x \leq 0 \]

\[ x = (x_1 + x_2) \]

\[ x = \frac{-b}{2a} \]

\[ x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

\[ x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

For all initial conditions, just bang bang.
\[ x(t) = e^{-t}(A + b) + (C + t) \]

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{-t} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

General solution for \( \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \) is

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Initial conditions:

\[ x(0) = 0 \]

Equations from

\[ x(0) = 0 \]

\[ x(t) = e^{-t}(A + b) + (C + t) \]

\[ x(0) = 0 \]

Quasilinearization
ON THE ITERATION SCHEME:

We can place a stop at a norm of norms between two solutions.

For example, when combining using $X$, solve for $X(t)$ and $\tilde{X}(t)$, etc., for

\[
(1) \quad X(t) = \begin{pmatrix}
X_1(t) \\
X_2(t)
\end{pmatrix}
\]

Same for $Y$

\[
\begin{pmatrix}
Y_1(t) \\
Y_2(t)
\end{pmatrix} = \begin{pmatrix}
od \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
X_1(t) \\
X_2(t)
\end{pmatrix} = \begin{pmatrix}
\frac{X_1^2(t)}{X_2(t)} + \frac{X_2^2(t)}{X_1(t)} \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
Y_1(t) \\
Y_2(t)
\end{pmatrix} = \begin{pmatrix}
\frac{Y_1^2(t)}{Y_2(t)} + \frac{Y_2^2(t)}{Y_1(t)} \\
0
\end{pmatrix}
\]

Let $X(0, t)$ be a known solution.

\[
X(t) = \begin{pmatrix}
X_1(t) \\
X_2(t)
\end{pmatrix}
\]

\[
X(0) = \begin{pmatrix}
X_1(0) \\
X_2(0)
\end{pmatrix}
\]

LINEARIZATION...

\[
\text{(Initial Guess)}
\]

\[
\text{(Linearization)}
\]
\[(0) X = (1) X (h - c) + (x X 2 - o X 2) = (1) X \]

\[(0) Y = (1) Y (-1) + (x Y 2 - o Y 2) = (1) Y \]

\[(0) Z = (1) Z (h - c) + (x Z 2 - o Z 2) = (1) Z \]

Next Step is Linearization:

\[N = 0 \]

\[Y = \frac{1}{2} x = \frac{1}{2} y = \frac{1}{2} z \]

Thus:

\[X = 0 \]

\[Y = 0 \]

\[Z = 0 \]
\[ y_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

\[ y_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]

\[ y_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \]

**Question 1:** How do we solve?

\[ \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

Let's look at the general vector case.

\[ y = A(t) \cdot y_1 + E(t) \cdot y_2 + E(t) \cdot y_3 \]

For vector case,

\[ y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} \]

\[ y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

\[ y(\infty) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

\[ y(t) = y_1(t) + y_2(t) + y_3(t) \]
\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_5 \\
x_6 \\
x_7 \\
x_8
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \\
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} &= \begin{bmatrix}
0 \\
1 \\
0 \\
1
\end{bmatrix}
\end{align*}
\]

In scalar case:

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_5 \\
x_6 \\
x_7 \\
x_8
\end{bmatrix} = \begin{bmatrix}
x_1 + x_2 \\
x_3 + x_4 \\
x_5 + x_6 \\
x_7 + x_8
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + x_1 \\
x_2 + x_2 \\
x_3 + x_3 \\
x_4 + x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + A_{11} \\
x_2 + A_{22} \\
x_3 + A_{33} \\
x_4 + A_{44}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + A_{11} \\
x_2 + A_{22} \\
x_3 + A_{33} \\
x_4 + A_{44}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + x_1 \\
x_2 + x_2 \\
x_3 + x_3 \\
x_4 + x_4
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + x_1 \\
x_2 + x_2 \\
x_3 + x_3 \\
x_4 + x_4
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + x_1 \\
x_2 + x_2 \\
x_3 + x_3 \\
x_4 + x_4
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} \\
&= \begin{bmatrix}
x_1 + x_1 \\
x_2 + x_2 \\
x_3 + x_3 \\
x_4 + x_4
\end{bmatrix}
\end{align*}
\]
Consider the more general vector case again.

\[
\begin{bmatrix}
X_1^n \\
X_2^n
\end{bmatrix} = \begin{bmatrix}
X_1^0 \\
X_2^0
\end{bmatrix} = A^n \begin{bmatrix}
X_1^0 \\
X_2^0
\end{bmatrix}
\]

such that

\[
\begin{bmatrix}
X_1^0 \\
X_2^0
\end{bmatrix} = \begin{bmatrix}
x_1 h_1(0) \\
x_2 h_1(0)
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_1^0 \\
X_2^0
\end{bmatrix} = \begin{bmatrix}
x_1 h_2(0) \\
x_2 h_2(0)
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix}
X_1^n \\
X_2^n
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
CONVERSION FORMULAS

OUR INITIAL TIME =

\[
C_0 = C_1 + C_2
\]

\[
X(0) = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
X(0) \\
X(1) \\
X(2) \\
X(n)
\end{bmatrix}
= \epsilon \begin{bmatrix}
1 & 0 & 0 & 0 \\
X_1 & 1 & 0 & 0 \\
X_2 & X_1 & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
X_0 \\
X_1 \\
X_2 \\
\vdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_0 \\
X_1 \\
X_2 \\
\vdots
\end{bmatrix}
= \epsilon \begin{bmatrix}
1 & 0 & 0 & 0 \\
X_1 & 1 & 0 & 0 \\
X_2 & X_1 & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}^{-1}
\begin{bmatrix}
X(0) \\
X(1) \\
X(2) \\
X(n)
\end{bmatrix}
\]

THUS

\[
\begin{bmatrix}
X_0 \\
X(0)
\end{bmatrix} = \begin{bmatrix}
0 \\
X_0
\end{bmatrix}
\]

\[
\int \begin{bmatrix}
X_0 \\
X_1 \\
X_2 \\
\vdots
\end{bmatrix} \, dx = \begin{bmatrix}
\int X_0 \, dx \\
\int X_1 \, dx \\
\int X_2 \, dx \\
\vdots
\end{bmatrix}
\]

ON TO PARTICULAR SECTION.
\[ u = L - B + 1 \]
\[ \Rightarrow \]
\[ v = L - B + 1 \]
\[ A(0) = 0 = A \]
\[ x + 1 \]
\[ 0 = x \]
\[ \frac{x}{3} \]
\[ x(x) = x(1) = 0 \]

**Solution**

\[ \frac{x^2}{2} + \frac{x}{2} \]

\[ x = 0 \]

\[ x = 0 \]

\[ c_1 \text{xe}^{-x} + c_2 \cos x + c_3 \sin x \]

**Test + 1 Solutions**

Complete Test.

And our solution is

\[ c = \sqrt{(c_1 + c_2 + c_3)^2} \]

Call \( c \) the max vector.

\[ A(x) = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{bmatrix} \]

Only unknowns \( m = c - c_0 \)

\[ A(f, \frac{\pi}{2}) = \begin{bmatrix} \cos \left( \frac{\pi}{2} \right) \\ \sin \left( \frac{\pi}{2} \right) \\ \cos \left( \frac{\pi}{2} \right) \end{bmatrix} \]

How about final time?
\[ E = \phi(x,t) = \phi(x^2 + t) \]

1. \( x = \pm \sqrt{t} \)
2. \( \phi(x,0) = \phi(x) \) at \( t = 0 \)
3. \( \phi(-x, t) = \phi(x, t) \) for all \( x \)

\[ x = \pm \sqrt{t} \] at \( t = 0 \)

Satisfy E FUNCTION: 

\[ x = \sqrt{t} \] at \( t = 0 \)
\[ J = \int_0^t \left[ \frac{d}{dt} (x(t) + b) \right] dt \]

Consider \( x(t) = x_i - \frac{\omega}{2} \mathbf{e} \mathbf{e}^T \mathbf{e} \mathbf{e} \cdot x_i \)

The unit vector in direction of \( x_i \) is \( \mathbf{e} \).

\( x_i = x(t_0) = x_i \)

\[ x = \frac{d}{dt} x_i + \theta(x_i) \]

\[ x_i + 1 = x_i - \frac{\omega}{2} \mathbf{e} \mathbf{e}^T \mathbf{e} \mathbf{e} \cdot x_i \]

\[ x_i = \frac{6 \mathbf{e}}{\mathbf{e}} \]

\[ \mathbf{e} = \frac{x_i}{\|x_i\|} \]
\[ S_u = \sqrt{V_u} \quad \text{(e)} \]

**Consider**

**We know** \( x, y, \text{ and } z \)

**Thus**

\[ 6.4 = \sqrt{24} \quad \text{and} \quad \text{not} \]

**Say applied on** \( x, y, \text{ and } z \)

\[ x = 5 \quad y = 3 \quad z = 4 \]

\[ T(t) = \frac{1}{2} x(t) \]

\[ T(t) = 0 \]

\[ (f) \]

\[ x = 0 \quad 4 \quad 5 \]

\[ \text{Thus} \]

\[ x(t) = \int_{0}^{t} f(s) \, ds \]

\[ \text{such that} \]

\[ 5 \times 5 = \frac{1}{3} x \]

\[ 5 \times 5 \quad 1 \quad 6 \quad 8 \quad 9 \]

\[ 5 \quad 5 \quad 1 \quad 6 \quad 8 \quad 9 \]

\[ \begin{align*}
 5 & = 1 + \frac{1}{3} x \\
 0 & = t
\end{align*} \]

\[ a = \frac{\dot{x}}{t} \]

\[ a = \frac{\dot{x}}{t} + \frac{1}{3} x \quad t \]

\[ \text{where} \]

\[ a = \frac{\dot{x}}{t} + \frac{x}{t} \quad t=
\]

\[ \text{and} \]

\[ \text{and} \]

\[ \text{and} \]
(iii) Form $x(t)$. $x(t) = x_0 \cos \theta t$.

(iv) Substitute $x_0$, $\theta$, and $t$. $x(t) = x_0 \cos \theta t$.

(v) Integrate $x(t)$ to find $v(t)$. $v(t) = -x_0 \theta \sin \theta t$.

(vi) Integrate $v(t)$ to find $y(t)$. $y(t) = -\frac{x_0 \theta^2}{2} \cos \theta t$.

Solve for $x(t)$ and $v(t)$ given initial conditions $x_0$ and $v_0$.

---

Equation: $0 = \int_{-\infty}^{\infty} f(c) \, dc$.

$T_{\text{end}} = 6.0 + \frac{6.0}{2} = 9.0$.

Then $T_{\text{end}} = 7.50 \Rightarrow \text{pick this}$.
$X(t) = \frac{5e^{-16t}}{16}$

$x = 0 + y = \frac{c}{2}$

$X = \frac{x}{t} - 5x + (x - x + 0)$

$A = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$P = x$ (1) + $\sqrt{1} + \frac{1}{2} + \frac{1}{2}$

$x = x + y - x + 0$

$P = x + 0$

Example

Start over again (c)

$v(t) = v_0 - t (\Delta c)$

(1) Calculate $v(t)$

If $\Delta t$ are not known, we're done.

Is $\Delta t$ known?

Compute $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

(2) Evaluate
\[ \begin{align*} 
\text{(i)} \quad & y(0) = 0 \quad \Rightarrow k = 0 \\
\text{(ii)} \quad & \dot{x}(t) = 2x(t) - 1 \quad \Rightarrow x(t) = x_0 + 2e^{-t} \\
\text{(iii)} \quad & x_0(t) = 2x(t) - 1 \quad \Rightarrow x_0(t) = x_0 + 2(2e^{-t} + 1) \\
\end{align*} \]
\[ x = f(x, y, t) \]

\[ \dot{x} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial t} \]

\[ \frac{dx}{dt} = -\frac{1}{t} \int_0^t (x(t') - f(t', y(t'))) dt' \]

\[ \frac{dy}{dt} = -\frac{2}{\beta} \int_0^t R(x(t') - y(t')) dt' \]

\[ \frac{d\theta}{dt} = \frac{1}{\theta} \int_0^t R(x(t') - y(t')) dt' \]

\[ \frac{d\xi}{dt} = \int_0^t R(x(t') - y(t')) dt' \]

\[ x(0) = 1, y(0) = 1, \theta(0) = 0, \xi(0) = 0 \]

We wanna estimate \( a \) to minimize \( Q \).

\[ Q = \frac{1}{2} \int_0^t \left( x(t) - f(t, y(t)) \right)^2 dt \]

\[ \text{Differential Approximation} \]
\[
\text{(5) } x - (5) \sqrt{\frac{9 + 3}{9}} = (5) \sqrt{\frac{2x^2}{1}} + x - 6v = 6 \cdot 6 \cdot 2x^2 = 6 \cdot 6 \cdot x - 6v
\]

Now we wanna find \( x \in \mathbb{C} \)

\[
\begin{bmatrix}
\frac{6u - 6x^2}{2} \\
\frac{2x^2 + 8x^2}{2}
\end{bmatrix} = \begin{bmatrix}
x^2 \\
x^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{6u - 6x^2}{2} \\
\frac{2x^2 + 8x^2}{2}
\end{bmatrix} = \sqrt{1}
\]

\[
\begin{bmatrix}
x^2 \\
x^2
\end{bmatrix} = \begin{bmatrix}
x \\
1
\end{bmatrix}
\]

\[
x = \int (x^2 + c) \quad 0 \leq c < 2
\]

\[\mathbb{E}
\]

\[
\text{Thus } \quad p \cdot \frac{x}{h} \cdot \sqrt{\frac{(x^2 - c)^2}{h}} = 0
\]

\[
\text{Thus } \quad p = 2, \quad x = -\sqrt{2c}
\]

\[\mathbb{E}
\]
\[ \frac{d}{dt} \left( \frac{v}{\varepsilon} \right) = \frac{v}{\varepsilon} R \frac{dv}{dt} \]

\[ y = 2x + x^2 \]

\[ x_1 = \frac{2}{5}(x_1 - x_2 + x_3) \]

Now consider term]

\[ x_2 = \frac{2}{5}(x_2 - x_1 + x_3) \]

\[ x_3 = \frac{2}{5}(x_3 - x_1 + x_2) \]
\[ \frac{d\gamma}{dt} = \frac{25}{3} x^2 - \frac{25}{6} x^3 \cdot x \cdot \frac{dt}{\gamma} - 4 \cdot (x^2 + E_0 x^2) \]

OR

\[ \int_{0}^{\gamma} \frac{d\gamma}{\gamma} = \int_{0}^{x_1} \frac{25}{3} x^2 - \frac{25}{6} x^3 \cdot x \cdot \frac{dt}{\gamma} - 4 \cdot (x^2 + E_0 x^2) \]

So, we get (assuming \( p = 1 \))

\[ \begin{bmatrix} 0 & \frac{5}{3} x_2 & 0 & 0 & 0 & 0 \\ \frac{5}{3} x_2 & \frac{5}{3} x_3 & \frac{5}{3} x_4 & \frac{5}{3} x_5 & \frac{5}{3} x_6 & \frac{5}{3} x_7 \\ 0 & 0 & \frac{5}{3} x_5 & \frac{5}{3} x_6 & \frac{5}{3} x_7 & \frac{5}{3} x_8 \\ \end{bmatrix} = 0 \]
\[
\begin{bmatrix}
\sqrt{9} & \sqrt{3} & \sqrt{1} & \sqrt{1} \\
\end{bmatrix}
= \begin{bmatrix}
\theta & 0 & 0 & 0 \\
\end{bmatrix}
\]

Which are like \( \frac{1}{2} \) of there \( \theta \) can solve for \( x_1 \) and \( x_2 \) we know \( x \) and \( x_2 \) we have

\[
\int_{-\infty}^{L} 2x\frac{e^{-x^2/2}}{\sqrt{2\pi}} dt = \int_{-\infty}^{L} 2x\frac{e^{-x^2/2}}{\sqrt{2\pi}} dt + \int_{L}^{\infty} 2x\frac{e^{-x^2/2}}{\sqrt{2\pi}} dt
\]

\[
\int_{-\infty}^{L} 2x\frac{e^{-x^2/2}}{\sqrt{2\pi}} dt = \int_{L}^{\infty} 2x\frac{e^{-x^2/2}}{\sqrt{2\pi}} dt
\]

And

\[
2p^{2x} \times \frac{1}{e^{\frac{L^2}{2}}} = \frac{e^{L^2}}{e^{\frac{L^2}{2}}}
\]

Gives
Coupling with Quasilinearization.

This problem with

We can in principle solve

\[ \langle c(t), x(t) \rangle = 0 \]

\[ c(t) \in L^2 \]

Assume we measure \( x \),

Give a total of new equations

\[ a \geq 0 \]

Since \( a \) is constant

\[ 0 \leq x \leq 1 \]

\[ x = f(x, 0, g(t)) \]

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THESE ASSUMING $X(t) = x_0$.

\[ 0 = \lambda \left( \frac{\dot{x}_0}{\phi} \right) + H + \frac{\dot{\lambda}}{\phi} \]

\[ \frac{\dot{\lambda}}{\phi} = \frac{\dot{x}_0}{\phi} - \frac{\dot{x}_0}{\phi} \]

\[ \frac{\lambda}{\phi} = 0 \]

\[ \lambda = 0 \]

\[ \dot{x}_0 = 0 \]

\[ \dot{\lambda} = 0 \]

\[ N \wedge + \Theta = \Theta \]

\[ \text{Hamiltonian} \]

\[ \mathcal{H} = \int_{t_0}^{t_1} \left( x'(t) \right)^2 + \phi(t) \]

\[ \phi(0) = c \]

\[ N(0) = m \]

\[ M(t) = \phi(0) \]

\[ \phi(t) = c \]

\[ \Theta = \Theta \]

\[ \dot{x}_0 = \dot{x}_0 \]

\[ \dot{\lambda} = \dot{\lambda} \]

\[ \text{The Bolza Problem} \]

\[ \text{TURNS out $A$ is CONSTANT} \]

\[ \Theta = \Theta \]

\[ \mathcal{H} = \mathcal{H} \]

\[ \text{Lagrangian} \]

\[ \text{Lagrange Multipliers} \]

\[ \text{Constraint Equations} \]

\[ \text{Transversality Conditions} \]

\[ \text{Corner Conditions} \]

\[ \text{Weierstrass-Erdman} \]

\[ \text{For when} \]

\[ \text{Performance Measure} \]

\[ \text{Variational Calculus} \]

\[ \text{Control} \]

\[ \text{Observability} \]

\[ \text{Control Observability} \]

\[ \text{Plug Sheet (Test 4)} \]
OPTIMAL CONTROL

H(x, y, t) = H(x', y', t')

PRINCIPLE OF MESS TRAVERSAL:

E = E(x, y, t) - E(x', y', t') - (x - x') \theta_x + (y - y') \theta_y > 0

WEIGHT FUNCTION:

0 = \Phi + \frac{x^2}{2\sigma^2} + \frac{y^2}{2\sigma^2} + \frac{t^2}{2\sigma^2}

\sigma = \frac{2}{P}

w = 0, w(t) = 0

\bar{v} = \frac{x^2}{\sigma^2} - \frac{y^2}{\sigma^2} + \frac{t^2}{\sigma^2}

Gives

LAE UN N

PROBLEM WITH INEQUILITU CONSTAINT

u = k x \Rightarrow k = KALMAN GAIN

F(t) = S

\dot{X} = F(x) + P R + A T P - \dot{P} R - E P + \dot{E} T P + \dot{G} = 0

\dot{X} = A x + B u

\dot{P} = F x + B T u

\dot{K} = X + P T u
Final CRM Sheet

\[ J = \int_{t_0}^{t_f} g(x(t), u(t)) \, dt \]

Pontryagin's Maximum Principle:

\[ J(x, u, t) = \int_{t_0}^{t_f} [L(x, u, t) + \lambda^T H(x, u, t)] \, dt \]

\[ \frac{d}{dt} \lambda^T = -M \]

Optimal Control:

\[ u^* = \arg \max_{u} J(x, u, t) \]

\[ E = g(x(t), u(t)) - \dot{\lambda}(x(t), u(t)) \]

\[ \frac{d}{dt} \lambda = -M - M \lambda + L \]

Boundary Conditions:

\[ \lambda(t_f) = 0 \]

Euler-Lagrange Equations:

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial u} = 0 \]

Constrained Minimization:

\[ \min_{u} J(x, u, t) \]

\[ g(x, u, t) = 0 \]

\[ M = \begin{bmatrix} \gamma & \delta \end{bmatrix} \]

\[ L = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \]

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

\[ g_1 = g_2 = \cdots = g_n = 0 \]

\[ \text{constraint: } \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0 \]

\[ \text{Euler's Equation} \]

\[ x(t) = x_0 + \int_{t_0}^{t} v(s) \, ds \]

\[ v(t) = \begin{cases} v_0 & t \leq t_0 \\ \frac{\partial L}{\partial x} & t > t_0 \end{cases} \]

\[ E = g(x(t), u(t)) - \dot{\lambda}(x(t), u(t)) \]

\[ \frac{d}{dt} \lambda = -M - M \lambda + L \]

\[ \lambda(t_f) = 0 \]

\[ J = \int_{t_0}^{t_f} g(x(t), u(t)) \, dt \]

\[ \text{Euler's Equation} \]

\[ x(t) = x_0 + \int_{t_0}^{t} v(s) \, ds \]

\[ v(t) = \begin{cases} v_0 & t \leq t_0 \\ \frac{\partial L}{\partial x} & t > t_0 \end{cases} \]

\[ E = g(x(t), u(t)) - \dot{\lambda}(x(t), u(t)) \]

\[ \frac{d}{dt} \lambda = -M - M \lambda + L \]

\[ \lambda(t_f) = 0 \]

\[ J = \int_{t_0}^{t_f} g(x(t), u(t)) \, dt \]

\[ \text{Euler's Equation} \]

\[ x(t) = x_0 + \int_{t_0}^{t} v(s) \, ds \]

\[ v(t) = \begin{cases} v_0 & t \leq t_0 \\ \frac{\partial L}{\partial x} & t > t_0 \end{cases} \]

\[ E = g(x(t), u(t)) - \dot{\lambda}(x(t), u(t)) \]

\[ \frac{d}{dt} \lambda = -M - M \lambda + L \]

\[ \lambda(t_f) = 0 \]
\[ \frac{dx}{dt} = x^3 + t \]

\[ J(x) = \int_0^1 (x^4 + t^2) \, dt \]

\[ x(0) = 0, \quad x(T) = 1 \]

**Eulerian Approximation**

1. Compute \( u(t) = v(t) - \varepsilon t \)

2. Compute \( z(t) = \frac{\partial u}{\partial t} \)

3. Compute \( t' = \int (x, x') (t) \) from \( x(0)^T \)

4. Integrate constant \( A(t) = \frac{\partial v}{\partial t} \)

5. Compute \( \frac{\partial x}{\partial t} \) (initial value \( C = 0 \))

6. Compute \( \frac{\partial x}{\partial u} \) (final value \( C = 0 \))

**Functional Estimates**

\[ \langle \frac{\partial x}{\partial t} \rangle = \int \langle \frac{\partial x}{\partial t} \rangle \, dt \]

\[ = \int (x, x') (t) \, dt \]

**Method of Steepest Descent**

\[ \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} \]

\[ x = 0 \]

Assume \( y(0) = 1 \)

\[ y(T) = 0 \]

\[ \begin{bmatrix} y(0) \\ y(T) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(T) \end{bmatrix} \]

**Quasilinearization**

For minimum fuel problem:

- **Bang-Bang Control**
- Singular problems
- Use \( y = 0 \) and plug away
Anncer
Elementary Nonlinear Electronic Circuits
Carlin & Giordano
Network Theory: An Introduction to Reciprocal and Nonreciprocal Circuits
Chirlian
Integrated and Active Network Analysis and Synthesis
Deutsch
Systems Analysis Techniques
Herrero & Willoner
Synthesis of Filters
Manasse, Eckert & Gray
Modern Transistor Electronics Analysis and Design
Newcomb
Active Integrated Circuit Synthesis
Sage
Optimum Systems Control
Van Valkenburg
Network Analysis, 2nd Ed.

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Many problems in modern system theory may be simply stated as extreme value problems. These can be resolved via the calculus of extrema which is the natural solution method whenever one desires to find parameter values which minimize or maximize a quantity dependent upon them. In this chapter we will consider several such problems, starting with simple scalar problems and concluding with a discussion of the vector case. The method of Lagrange multipliers will be introduced and used to solve constrained extrema problems for single-stage decision processes. A brief discussion of linear and nonlinear programming will be presented. Multistage decision processes, which can be treated by the calculus of extrema, will be reserved for a variational treatment which will result in a discrete maximum principle. Much of the work in this chapter is very basic, and a selection of only references [1] through [5] of direct interest to the systems control area is given.

2.1 Maxima and minima (scalar process)

A real function $f(x)$, defined for a scalar $x = \alpha$, has a relative maximum or a relative minimum $f(\alpha)$ for $x = \alpha$ if and only if there exists a positive real number $\delta$ such that, respectively,

$$\Delta f = f(\alpha + \Delta x) - f(\alpha) < 0 \quad (2.1-1)$$
\[
\Delta f = f(\alpha + \Delta x) - f(\alpha) > 0
\]

for all \( \Delta x = x - \alpha \) such that \( f(\alpha + \Delta x) \) exists and \( 0 < |\Delta x| < \delta \). Further, if \( df(x)/dx \) exists and is also continuous at \( x = \alpha \), then \( f(\alpha) \) can be an interior maximum or minimum only if

\[
\frac{df(x)}{dx} \bigg|_{x=\alpha} = 0
\]

If \( f(x) \) has a continuous second derivative for \( x = \alpha \), the nature of the extremum at \( x = \alpha \) can be determined. The following well-known procedure can be used for the determination of the extrema of a given function \( y = f(x) \).

1. Differentiate \( y \) with respect to \( x \).
2. For each value of \( x \), determine the specific values of \( \alpha \) which satisfy the equation \( dy/dx = 0 \).
3. Test to see what kind of extremum the function has for each value of \( \alpha \) thus obtained. This we can easily accomplish by the second-derivative test in which we substitute each value of \( \alpha \) into the second derivative of \( y \) with respect to \( x \) and apply the following rule:

\[
\begin{align*}
&\frac{d^2y}{dx^2} > 0 \quad \text{then } y \text{ has a relative minimum} \\
&\frac{d^2y}{dx^2} < 0 \quad \text{then } y \text{ has a relative maximum} \\
&\frac{d^2y}{dx^2} = 0 \quad \text{then } y \text{ has a stationary point}
\end{align*}
\]

4. Evaluate the actual value of the extrema by substituting each value of \( \alpha \) obtained into \( f(x) \).

There are three different types of extrema possible. If a value of \( \alpha \) can be found such that \( f(\alpha) \) is an extremum for all \( x \) throughout its domain of definition, \( f(x) \) is said to have an absolute extremum. If a value of \( \alpha \) can be found such that \( f(\alpha) \) has an extremum throughout a bounded neighborhood of \( x \), \( f(x) \) has a relative extremum at \( x = \alpha \). If \( f(x) \) is defined only for a limited range of values of \( x \), and if \( f(x) \) has an extremum at either boundary of \( x \) (with respect to all the values \( f(x) \) has for all values of \( x \) contained within the limited range of \( x \)), then \( f(x) \) has an extremum at its boundary. These different types of extrema are illustrated in Fig. 2.1-1. We will have opportunity to apply these concepts to parameter optimization of control systems in Sections 8.2 and 13.3-1.

### 2.2 Extrema of functions of two or more variables

The extrema-finding technique can be extended to include functions of more than one variable. Suppose \( y = f(x_1, x_2, \ldots, x_n) = f(x) \). A procedure similar to the previous one is used, using partial derivatives instead of total derivatives. A simple example will illustrate the procedure to be followed.

**Example 2.2-1**

Let us consider the maximization of

\[
y(x) = \frac{1}{(x_1 - 1)^2 + (x_2 - 1)^2 + 1}, \quad x^r = [x_1, x_2]
\]
where $x^T$ is to indicate transpose of the column vector $x$.† Following an extended version of the foregoing scalar procedure, we take the partial derivatives of $y$ with respect to $x_1$ and $x_2$ and set them equal to zero to obtain:

$$
\frac{\partial y}{\partial x_1} = \frac{(-1)(2x_2 - 2)}{[(x_1 + 1)^2 + (x_2 - 1)^2 + 1]^2} = 0, \quad \alpha_1 = 1
$$

$$
\frac{\partial y}{\partial x_2} = \frac{(-1)(2x_1 - 2)}{[(x_1 - 1)^2 + (x_2 - 1)^2 + 1]^2} = 0, \quad \alpha_2 = 1
$$

Thus, since $\alpha_1 = \alpha_2 = 1$ are the only extrema, and since a simple computation shows that the second derivatives are nonpositive at this extrema, we see that we have a maximum at the point $x^T = [1, 1]$.

**Example 2.2-2**

Let us now suppose that the allowable range of $x$ is constrained such that $|x_1| \leq \frac{1}{2}$ and $|x_2| \leq \frac{1}{2}$. It is desired to find the value of $x$ which yields a maximum for the $y = f(x)$ of Example 2.2-1 in the allowable or admissible range of $x$.

This region of state space is also shown in Fig. 2.2-1. From this figure, it is apparent that, for this simple problem, $y = f(x)$ has an extremum (maximum) somewhere on the boundary of the admissible range for $x$, in fact precisely at $x^T = [\frac{1}{2}, \frac{1}{2}]$. This is a very simple example of optimization with an inequality constraint. We will have considerably more to say about this very important type of constraint when we consider dynamic systems and the calculus of variations.

**Example 2.2-3**

A slightly more difficult problem arises if the allowable range of $x$ is constrained such that the Euclidean norm of $x$ equals one. Symbolically, this means that $||x||^2 = x^T x = x_1^2 + x_2^2 + \ldots + x_n^2 = \langle x, x \rangle$. Since the dimension of the example that we are considering is two, the Euclidean norm squared becomes $||x||^2 = x_1^2 + x_2^2$.

One approach to the problem is to solve for $x_1$ in terms of $x_2$, then solve for $y = f(x)$ in terms of $x_2$ alone. This will then allow us to use the standard scalar procedure. From the given constraint on the length of the Euclidean norm, we have $x_1 = (1 - x_2)^{1/2}$. Substituting this into the expression for $y(x)$ of Example 2.2-1, we find that

$$
 y(x_2) = \frac{1}{(\pm \sqrt{1 - x_2^2} - 1)^2 + (x_2 - 1)^2 + 1}
$$

where $y(x_2)$ has the given constraint imbedded into it. The next step is to differentiate this expression with respect to the remaining variable, $x_2$, and set the result equal to zero. This yields two solutions. The second-derivative test shows that a maximum (which is easily shown to be an absolute maximum) occurs at $x^T = [0.707, 0.707]$ and that an (absolute) minimum occurs at $x^T = [-0.707, -0.707]$.

We note that, in the absence of the equality constraint, this problem has no

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† Appendix A contains a brief presentation of vector matrix notations and vector matrix calculus.
of this type, namely \( \|x\|^2 \leq 1 \) and \( \|x\|^2 < 1 \). The first constraint set is closed (and convex) since it includes the boundary \( \|x\|^2 = x^2 + y^2 = 1 \). The second is open (and convex) since it does not include the boundary. It is generally quite difficult to work with constraints of this form. One method, satisfactory in quite a few problems, is to ignore the constraint and find the maximum (or minimum). If this turns out to be interior to the boundary of the constraint set, we have the solution. If the maximum (or minimum) occurs outside the boundary, the inequality constraint is treated as an equality constraint, and a solution is found with this constraint. Another method, to be discussed later, is to convert the inequality constraint to an equality constraint. Figure 2.2-1 illustrates salient features of these examples.

2.3 Constrained extrema problems—Lagrange multipliers

An alternate approach to extremizing a function (i.e., find those values of the independent variables which cause the dependent variable to have an extremum) with given constraints or accessory conditions is to make appropriate adjustments on the independent variable by using an adjustable multiplying parameter, commonly called a Lagrange multiplier. The procedure is to form a new function by adjoining the given constraint to the original function. This new function, then, is extremized, by means of the previously developed method. We will solve an example first by the more straightforward, but often more cumbersome, procedure and then by using the Lagrange multiplier. Considerably more justification for the Lagrange multiplier procedure will be provided in the next chapter on variational calculus.

Example 2.3-1

A tin can manufacturer wants to maximize the volume of a certain run of cans subject to the constraint that the area of tin used be a given constant. If a fixed metal thickness is assumed, a volume of tin constraint implies that the cross-sectional area is constrained.

The defining equations for this problem are:

- Volume: \( V(r, l) = \pi rl \)  \hspace{1cm} (1)
- Cross-sectional area: \( A(r, l) = 2\pi r^2 + 2\pi rl = A_o \)  \hspace{1cm} (2)

Our problem is to maximize \( V(r, l) \) subject to keeping \( A(r, l) = A_o \), where \( A_o \) is a given constant. The same approach can be used here as in Example 2.2-3. We solve for \( l \) in terms of \( r \) (or if preferred, \( r \) in terms of \( l \)) and then express the volume as a function of \( r \) alone, noting that the constraint on the cross-sectional area is now imbedded into the expression for the volume. We then examine the first and second derivatives to discern the character and location of the extrema.

Method 1

From Eq. (2) we have

\[
l = \frac{A_o - 2\pi r^2}{2\pi} \tag{3}\]

By substituting Eq. (3) into Eq. (1), we obtain

\[
V(r) = \frac{1}{2} A_o - \pi \sqrt{A_o} r^2 = \pi \cdot \frac{\sqrt{A_o}}{2} \tag{4}\]

We differentiate \( V \) with respect to \( r \) and set the result equal to zero to obtain

\[
dV(r) = \frac{dV}{dr} = A_o - 3\pi r^2 = 0, \quad r = \sqrt{\frac{A_o}{3\pi}} \tag{5}\]

We now substitute Eq. (5) into Eq. (2) and solve for \( l \):

\[
l = \sqrt{\frac{2A_o}{3\pi}} \tag{6}\]

It is interesting to obtain the optimum length-to-radius ratio. In doing this, we see that, to get maximum volume, we make the length of the tin can equal the diameter, keeping cross-sectional area equal to a given constant.

Method 2

By using the Lagrange multiplier, we again want to extremize (maximize) the volume \( V(r, l) \) subject to the constraint \( A(r, l) = A_o \). First we form the adjoined function

\[
V'(r, l) = V(r, l) + \lambda (A(r, l) - A_o) \]

where \( \lambda \) is the Lagrange multiplier. In terms of the parameters of the tin can, this expression becomes

\[
V'(r, l) = \pi rl + \lambda [2\pi r^2 + 2\pi rl - A_o] \]

We take the first partial derivative with respect to each of the variables and set each result equal to zero. Thus we obtain

\[
\frac{\partial V'(r, l)}{\partial l} = \pi r + \lambda [4\pi r + 2\pi l] = 0, \quad l = 2r \tag{7}\]

We now evaluate \( \lambda \) subject to given constraint, \( A(r, l) = A_o \) or

\[
A_o = 2\pi r^2 + 2\pi rl \]

In terms of the obtained values of \( r \) and \( l \), this becomes

\[
A_o = 2\pi(4\lambda^2) + 2\pi(-2\lambda)(-4\lambda) \]

so

\[
\lambda = \pm \sqrt{\frac{A_o}{24\pi}} \]
\[ r = 2\sqrt{\frac{A_R}{24\pi}}, \quad l = 4\sqrt{\frac{A_R}{24\pi}} \]

We note that the negative square root is selected for \( r \) to make \( r \) and \( l \) physically realizable quantities. We further note that the length-to-radius ratio is the same as obtained by the first method, as it well should be.

### 2.4 Vector formulation of extrema problems—single-stage decision processes

Considerable notational simplification occurs if functions of more than one variable are written in state vector notation. Thus a scalar function of several variables which is to be extremized

\[ J = \theta(x_1, x_2, \ldots, x_n) \tag{2.4-1} \]

may be written as

\[ J = \theta(x) \tag{2.4-2} \]

where

\[ x^T = [x_1, x_2, \ldots, x_n] \tag{2.4-3} \]

For the majority of systems problems, it is convenient to distinguish between control vectors and state vectors. We generally desire to find a control vector, \( u \) or \( u(k) \), or \( u(t) \) if we have a multistage or continuous process which minimizes or maximizes some scalar index of performance of the system. This performance index will be called \( J \). Possibly the simplest single-stage decision process with equality constraints is to minimize or maximize the scalar index of performance

\[ J = \theta(x, u) \tag{2.4-4} \]

subject to the equality constraint

\[ f(x, u) = 0 \tag{2.4-5} \]

where \( x \) is an \( n \) vector

\[ x^T = [x_1, x_2, \ldots, x_n] \tag{2.4-6} \]

\( u \) is an \( m \) vector

\[ u^T = [u_1, u_2, \ldots, u_m] \tag{2.4-7} \]

\( f \) is an \( n \) vector function

\[ f^T(x, u) = [f_1(x, u), f_2(x, u), \ldots, f_n(x, u)] \tag{2.4-8} \]

The solution proceeds as follows. We adjoin Eq. (2.4-5) to Eq. (2.4-4) with a vector Lagrange multiplier in order to form a scalar entity which we will call \( H(x, u, \lambda) \):

\[ H(x, u, \lambda) = \theta(x, u) + \lambda^T f(x, u) \tag{2.4-9} \]

\[ \lambda^T = [\lambda_1, \lambda_2, \ldots, \lambda_n] \tag{2.4-10} \]

We now adjust \( x \) and \( u \) such that \( H \) is a maximum or minimum. This requires

\[ \frac{\partial H}{\partial x} = \frac{\partial f}{\partial x} + \lambda f^T(x, u) = 0 \tag{2.4-11} \]

\[ \frac{\partial H}{\partial u} = \frac{\partial f^T}{\partial u} + \frac{\partial f}{\partial u} \lambda = 0 \tag{2.4-12} \]

where

\[ \begin{bmatrix} \frac{\partial H}{\partial u} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial u_2}, \ldots, \frac{\partial H}{\partial u_m} \end{bmatrix} \tag{2.4-13} \]

Thus \( \partial H/\partial u \) may be interpreted as the gradient of \( H \) with respect to \( u \), which is commonly designated \( \nabla_u H \). Also,

\[ \frac{\partial f^T(x, u)}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_n}{\partial x_1} \\ \vdots \\ \frac{\partial f_1}{\partial x_n}, \frac{\partial f_2}{\partial x_n}, \ldots, \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{2.4-14} \]

It should be noted that Eq. (2.4-14) is similar to the transpose of the Jacobian of a vector

\[ [J_f(x, u)]^T = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \ldots, \frac{\partial f_n}{\partial x_1} \\ \vdots \\ \frac{\partial f_1}{\partial x_n}, \frac{\partial f_2}{\partial x_n}, \ldots, \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{2.4-15} \]

with at least two important differences: \( \partial f(x, u)/\partial u \) need not be square and is a matrix rather than a determinant. In order that \( J \) be an extremum, not only must

\[ \frac{\partial H}{\partial x} = 0; \quad \frac{\partial H}{\partial u} = 0 \tag{2.4-16} \]

but also the second variation of \( H \) must be greater than zero for a minimum or less than zero for a maximum (see second-derivative test, Section 2.1.)

\[ \dagger \text{This scalar quantity, the Hamiltonian, has a number of very interesting properties that will be mentioned in later chapters.} \]
Chapters 3, ad 13 will provide us with considerably more information on the second variation than we present here. To see what this constraint on the second variation of \( H \) means, in terms of the necessary conditions required for making \( J(x, u) \) have an extremum, let us now formulate the second variation of \( H(x, u, \lambda) \). The first variation of \( H(x, u, \lambda) \) is

\[
\delta H = \left( \frac{\partial H}{\partial x} \right)^T \delta x + \left( \frac{\partial H}{\partial u} \right)^T \delta u
\]

which is the linear part of

\[
\Delta H = H(x + \delta x, u + \delta u) - H(x, u) \tag{2.4-18}
\]

To get the second variation of \( H \), denoted \( \delta^2 H \), we take the second-order part of the expansion of Eq. (2.4-18) in a Taylor series about \( \delta u = 0 \), \( \delta x = 0 \) to obtain

\[
\delta^2 H = \frac{1}{2} \delta x^T \left[ \frac{\partial^2 H}{\partial x \partial x} \right] \delta x + \frac{1}{2} \delta u^T \left[ \frac{\partial^2 H}{\partial u \partial u} \right] \delta u \tag{2.4-19}
\]

In more compact notation, this becomes

\[
\delta^2 H = \frac{1}{2} [\delta x^T \delta u]^T \left[ \frac{\partial^2 H}{\partial x \partial x} \quad \frac{\partial^2 H}{\partial u \partial x} \right] \left( \frac{\partial^2 H}{\partial x \partial u} \quad \frac{\partial^2 H}{\partial u \partial u} \right) \delta x \delta u \tag{2.4-20}
\]

If we define

\[
\delta z^T = [\delta x^T \delta u]^T \quad \mathbf{P} = \left[ \begin{array}{cc} \frac{\partial}{\partial x} H & \frac{\partial}{\partial u} H \\ \frac{\partial}{\partial u} H & \frac{\partial}{\partial u} H \end{array} \right]
\]

Eq. (2.4-20) reduces to

\[
\delta^2 H = \frac{1}{2} \delta z^T \mathbf{P} \delta z = \frac{1}{2}(\delta z)^T \mathbf{P} \delta z \tag{2.4-22}
\]

which is recognized as the standard quadratic form. A positive definite quadratic form is defined as one for which \( \delta z^T \mathbf{P} \delta z > 0 \) for all nonzero \( \delta z \). A positive semidefinite matrix, \( \mathbf{P} \), is defined as one which has the property that \( \delta z^T \mathbf{P} \delta z \geq 0 \) for all nonzero \( \delta z \). In a similar fashion, negative definite and negative semidefinite quadratic forms and matrices are defined. Section 1.23 of Appendix A delineates a method which we can use to discern positive definiteness of a square matrix. Thus we can state the two necessary conditions for \( J(x, u) \) to have an extremum in a given interval of \( x \) for convex or concave \( J(x, u) \). If \( J(x, u) \) is not convex or concave, the second condition is only sufficient, and a quantity known as the bordered Hessian must be used to obtain the second necessary condition.

\[
\text{I. The following vectors are zero:}
\]

\[
\frac{\partial H}{\partial x} = 0; \quad \frac{\partial H}{\partial u} = 0
\]

\[
\text{II. The following matrix}
\]

\[
\left[ \begin{array}{cc} \frac{\partial}{\partial x} H & \frac{\partial}{\partial u} H \\ \frac{\partial}{\partial u} H & \frac{\partial}{\partial u} H \end{array} \right] = \begin{cases} \text{positive semidefinite for a minimum along } f(x, u) = 0 \\ \text{negative semidefinite for a maximum along } f(x, u) = 0 \end{cases}
\]

A sufficient condition for a function to have a minimum (maximum) given that the first variation vanishes is that the second variation be positive (negative) where the first variation vanishes [4]. These conditions are general and need be modified only if the possibility of a singular solution exists.

**Example 2.4-1**

Suppose that we have a linear system represented by

\[
f(x, u) = Ax + Bu + c = 0
\]

and wish to find the \( m \) vector \( u \) which minimizes

\[
J(x, u) = \frac{1}{2} \| u \|_1 + \frac{1}{2} \| x \|_1
\]

where \( A \) is an \( n \times n \) matrix, \( B \) is an \( n \times m \) matrix, \( x \), \( c \), and \( u \) are \( n \) vectors, \( R \) and \( Q \) are positive definite symmetric matrices of dimensionality \( m \times m \) and \( n \times n \).

The Hamiltonian function is formed by adjoining the cost function to the given constraint via the Lagrange multiplier technique which gives us

\[
H = \frac{1}{2} u^T R u + \frac{1}{2} x^T Q x + \lambda^T [Ax + Bu + c]
\]

In order to minimize \( J \), it is necessary that

\[
\frac{\partial H}{\partial x} = Q x + A^T \lambda = 0, \quad \frac{\partial H}{\partial u} = Ru + B^T \lambda = 0
\]

where \( \lambda \) is to be adjusted so that the given equality constraint is satisfied, or

\[
\frac{\partial H}{\partial x} = R \lambda - A^T \lambda = 0
\]

Thus we find that \( u = -K^{-1} A^T (A^T Q A + B^T R B)^{-1} c \). Thus we can determine whether

\[
\lambda = -K^{-1} A^T (A^T Q A + B^T R B)^{-1} c
\]

is the optimum \( u \) vector. We notice that it is necessary that the inverse of \( A \) exist in order for the \( u \) vector to exist. To check if this solution does in fact cause \( J(x, u) \) to have a minimum, we find the second variation and check the necessary condition II given earlier. From Eq. (2.4-19) and the specifications for this problem, we have

\[
\delta^2 J = \frac{1}{2} [\delta x^T \delta u]^T \left[ \begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right] \left( \delta x^T \quad \delta u^T \right) = \frac{1}{2} \delta x^T Q \delta x + \frac{1}{2} \delta u^T R \delta u
\]
For $J(x, u)$ we a minimum, $\delta^2 J \geq 0$, therefore $Q$ and $R$ must be non-negative definite. Since this is given in the statement of the problem, the solution, if it exists, does minimize $J(x, u)$. A further requirement is obtained by noting that the first variation of $J(x, u) = 0$ yields $A x + B u = 0$ and $\dot{x} = f(x, u)$, which is the well-known expression for the average of a number of observations.

Another interesting case occurs when we have computed $x$ for $r$ measurements and someone gives us an additional measurement. A great deal of effort would be involved in multiplying and inverting $H^T R^{-1} H$ if $H$ is, say, a 1000 by 20 matrix. To repeat this procedure for a new 1001 by 20 matrix would probably be prohibitive of computer time, particularly if “on-line” computation is a requirement. We are thus led to seek a solution which allows us to add the new measurement without repeating the entire calculation. A method which allows us to do this is called a recursive or sequential estimation scheme. Such schemes are of considerable importance in modern system theory and will be explored in much more detail in Chapters 10 and 15.

Assume a set of measurements is represented by

$$
\begin{align*}
\mathbf{z} &= H\mathbf{x} + \mathbf{v} \\
\mathbf{z} &= \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \quad H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{mn} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}
\end{align*}
$$

where $\mathbf{z}$ is given by $(H^T R^{-1} H)^{-1} H^T R^{-1} \mathbf{z}$. Now suppose that we obtain an additional measurement such that we have

$$
\begin{align*}
\mathbf{z} &= H\mathbf{x} + \mathbf{v} \\
\mathbf{z} &= \mathbf{x}_{m+1} + \Delta \mathbf{x} + \mathbf{v}
\end{align*}
$$

The problem now becomes one of obtaining the best estimate of $\mathbf{x}$, $\mathbf{x}_{m+1}$, such that

$$
J = \frac{1}{2} \left\| \begin{bmatrix} \mathbf{z} \\ \mathbf{x}_{m+1} \end{bmatrix} - \begin{bmatrix} H \\ \mathbf{h} \end{bmatrix}_m \right\|^2
$$

is minimum. Following a procedure similar to the previous one, we find the best estimate of $\mathbf{x}$ is

$$
\mathbf{x}_{m+1} = \left( H^T R^{-1} H \right)^{-1} H^T R^{-1} \mathbf{z}
$$

where for convenience we will now assume that the matrix $R$ is an identity matrix. This amounts to placing equal weight on each measurement. A recursive scheme may be developed by the use of the matrix inversion lemma [2, 3]. We recall that

$$
\begin{bmatrix} H^T & H \\ H^T & h^T \end{bmatrix}^{-1} = \begin{bmatrix} H^T H + hh^T \end{bmatrix}^{-1}
$$

If we define

$$
P_m^{-1} = H^T H, \quad P_{m+1}^{-1} = \begin{bmatrix} H^T & H \\ H^T & h^T \end{bmatrix}^{-1} = P_m^{-1} + hh^T
$$
then the $r$ inversion lemma

$$P_{n+1} = P_n - P_n h^T P_n h + [1 - h^T P_n h]$$

which will be developed in Section 10.4.1 in a more general form, yields for the recursion formula

$$s_{n+1} = P_m h^T p_n h + [1 - h^T P_m h]$$

Thus the new estimate is equal to the old plus a linear correction term based on the new data and the old $P_m$ only. For $m$ estimates of a scalar $x$ with $H$ as a unit vector of dimension $m$, we have

$$P_n^{-1} = m, \quad P_{n+1} = \frac{1}{m+1}, \quad x_m = \frac{1}{m} \sum_i x_i$$

$$s_{n+1} = s_n + \frac{1}{m+1}[s_n - s_m] = s_{m-1} + \frac{1}{m+1}$$

which is, of course, the expected answer in this simple case.

### 2.5 Linear and nonlinear programming

The previous section contains several examples of what are commonly called nonlinear programming problems. Basically, the nonlinear programming problem is concerned with the extremization of a continuous differentiable function of $n$ nonnegative variables $\theta(x_1, x_2, \ldots, x_n) = \theta(x)$ subject to $m$ inequality constraints $\Lambda(x) \leq 0, \ i = 1, 2, \ldots, m$. Figure 2.5-1 illustrates some basic ideas in a nonlinear programming problem. In nonlinear programming, the $\theta$ function is called an objective function—the function to be extremized. In this book we will commonly call such functions cost functions.

As we have seen, ordinary calculus methods may be used to find the extremum of unconstrained functions. If ordinary calculus is applied to extremize $\theta$, and if the resulting optimum vector $x$ lies entirely within the constraint set $\Lambda = 0$, and if $x_i \geq 0$, then that value of $x$ solves the optimization problem with the constraint. We have seen examples of this in Section 2.2 and Example 2.4-2. If the optimum value of $x$ computed by extremizing $\theta$ is outside the constraint set $\Lambda \leq 0$ then the optimum value of $x$ lies on the boundary of the constraint set. If we knew which one of the $m$ constraints $\Lambda$ determined the optimum, then we could apply the Lagrange multiplier method and use an equality sign for that particular constraint and ignore the other constraints since the optimum $x$ will be on the boundary of one of the known $m$ inequality constraints. In general, we find it necessary to exploit each of the inequality constraints to determine which one of the inequality constraints to use. It is possible that more than one of the $m$ inequality constraints will determine the optimum $x$ as illustrated in Fig. 2.5-1. We should remark that, in the typical nonlinear programming problem, the functions $\Lambda$ are convex, which insures that the possible region for an optimum
A special case of the nonlinear programming problem is the linear programming problem which occurs when the \( \theta \) and \( \Lambda \) functions are linear in the \( n \) vector \( x \). In this case we are assured that the optimum value of \( x \) lies on the boundary of two or more elements of the linear constraint set \( \Lambda(x) \leq 0 \). Clearly, the major problem is to decide which ones. This is a statement of the general linear programming problem. Of several methods available for solving the problem, the most used method appears to be the simplex method [5]. In order to use the method, certain restrictions must be applied. The variables \( x_i \) must be nonnegative, the constraints \( \Lambda_i \) must be linear equalities, and the cost function must be minimized by the optimum \( x \).

We may transform the general problem of linear programming, that of maximizing the cost function (objective function)

\[
J = n^T x
\]

with the \( m \) inequality constraints

\[
Bx \leq c
\]

into the restrictive form for the simplex method. Any number can be written as the difference of two nonnegative numbers. For instance, if \( x_i \) has no restrictions on its sign, we may let

\[
x_{n+1} - x_{n+2} = x_i, \quad x_{n+1} \geq 0, \quad x_{n+2} \geq 0
\]

This insures the nonnegativity of the variables. Unfortunately, every substitution of this type replaces one variable (\( x_i \)) by two variables (\( x_{n+1} \) and \( x_{n+2} \)). If the original problem formulation contains inequality constraints, we convert them to equality constraints by the introduction of nonnegative slack variables. For example, if we had the constraints

\[
2x_1 + 4x_2 + x_3 \geq 5, \quad 6x_1 + x_2 + x_3 \leq 4
\]

we would introduce the nonnegative variables \( x_4 \) and \( x_5 \) to obtain equalities

\[
2x_1 + 4x_2 + x_3 - x_4 = 5, \quad 6x_1 + x_2 + x_3 + x_5 = 4
\]

The variables \( x_4 \) and \( x_5 \) "take up the slack" in the inequalities and are called slack variables. Again, we increase the total number of variables to be considered. The linear programming problem may now be solved by the simplex method.

Since we are to be much more concerned with optimization in dynamic systems than static optimization, we will not develop the many theorems of linear and nonlinear programming. References [4] and [5] contain thorough discussions of both these topics. We will consider numerical methods for the optimization of single-stage decision processes in Section 13.3-1.

The extrema-finding techniques of this chapter, although quite sufficient for many different situations, will not, in general, allow the solution to many problems associated with control systems. Whereas the previously discussed techniques deal with methods for extremizing functions of one or several independent variables, in control-system design, we are typically concerned with extremizing certain types of functions whose independent variables are actually other functions. This type of function is called a functional. Although, as we might expect, many of the basic approaches for extremizing functionals are similar to those for extremizing functions, the end results are sometimes quite different. The solution to a given problem in extremizing a given function of one variable is, perhaps, a number associated with a coordinate point, while the analogous solution to a functional problem is a number associated with a function. The body of mathematics developed for extremizing functionals is variational calculus. This subject is at the very heart of optimal control theory and is a subject that we will explore in some detail throughout the remainder of this text.

**REFERENCES**


**PROBLEMS**

1. Find \( u \) such that

\[
J = x^T + u^T
\]

is minimized subject to the equation

\[
xu = 1
\]

Use the Lagrange multiplier technique as well as the basic method.

2. Discuss the singular solution problem where \( x \) is a two vector.

3. Find \( x_0 \) for a set of measurements where \( z = Hx \), where
4. Now suppose that an additional measurement
\[ z_t = 3.0; \quad h^T = [1, 1] \]
is taken. Compute \( \hat{x} \) by the smoothing method and the matrix inversion lemma method. Compare the effort involved via each method.

5. Verify the matrix inversion lemma if
\[ P_{t+1} = P_t + h_1'h_1P_t^{-1} \]
by showing that
\[ P_{t+1}^{-1}P_{t+1} = 1 \]

6. From Eqs. (2.4-14) and (2.4-17) calculate the third variation of \( H \) as given in Eq. (2.4-9).

7. Find the maximum value of
\[ \theta(x) = x_1^2 + x_2 \quad x_1 \geq 0, \quad x_2 \geq 0 \]
subject to the inequality constraints
\[ (x_1 - 4)^2 + x_2^2 \leq 1 \]
\[ (x_1 - 1)^2 + x_2^2 \leq 4 \]

8. Find the maximum value of
\[ J = x_1 + x_2, \quad x_1 \geq 0, \quad x_2 \geq 0 \]
subject to the constraints
\[ x_1 + \frac{1}{2}x_2 \leq 1 \]
\[ \frac{1}{2}x_1 + x_2 \leq 2 \]

9. Two alternate expressions were developed for the optimum \( u \) vector of Example (2.4.1). Show that the two expressions are equivalent and that the first solution will be easier to implement computationally if the dimension of \( u \) is lower than that of \( x \).

3.1 Dynamic optimization without constraints

We will now examine a functional of the simple form where \( t_e \) and \( t_f \) are fixed
\[ J(x) = \int_{t_e}^{t_f} \phi(x(t), \dot{x}(t), t) \, dt \quad (3.1-1) \]
Problems of time-minimization of this functional form are sometimes called Lagrange problems. These include the Bolza problem

\[ J(x) = \theta[x(t), t] \bigg|_{t_0}^{t_f} + \int_{t_0}^{t_f} \phi[x(t), \dot{x}(t), t] \, dt \]  

(3.1-2)

The inclusion is apparent if Eq. (3.1-2) is rewritten in the form

\[ J(x) = \int_{t_0}^{t_f} \Lambda[x(t), \dot{x}(t), t] \, dt \]  

(3.1-3)

where

\[ \Lambda[x(t), \dot{x}(t), t] = \phi[x(t), \dot{x}(t), t] + \frac{d}{dt} \theta[x(t), t] \]  

(3.1-4)

We would now like to find an \( x(t) \) such that the given \( J(x) \) is extremized (i.e., maximized or minimized, depending on the given physical problem). This \( x(t) \) is called an extremal, and only an extremal can cause \( J(x) \) to have an extremum. We will assume that we know the correct extremal curve; denoted \( x(t) \).

Thus we can write the expression (3.1-5) for a family of curves, starting at \( t = t_0 \) and ending at \( t = t_f \), which includes the extremal curve \( x(t) \).

\[ x(t) = x(t) + \epsilon \eta(t) \]  

(3.1-5)

where \( \eta(t) \) is a variation in \( x(t) \) and \( \epsilon \) is a small number. A plot of \( J(x) \) versus \( \epsilon \) for various choices of \( \eta(t) \) might appear as shown in Fig. 3.1-1. It is obvious that at \( \epsilon = 0 \), all curves are minimum since

\[ J(x) = J(x) \bigg|_{\epsilon = 0} \]  

(3.1-6)

Thus on the extremals we have

\[ \frac{\partial J(x)}{\partial \epsilon} \bigg|_{\epsilon = 0} = 0 \]  

(3.1-7)

independent of the value of \( \eta(t) \) chosen. Strictly speaking, the solution obtained from Eq. (3.1-5) could cause \( J(x) \) to have a maximum or minimum or be a stationary point. The condition for a minimum is that \( \frac{\partial J}{\partial \epsilon} \) be positive at \( \epsilon = 0 \) independent of \( \eta(t) \). However, in most physical problems, it is apparent that if a solution to Eq. (3.1-7) exists, it will be a solution which minimizes (maximizes) the integral, \( J(x) \), as desired. Now we can extremize Eq. (3.1-1) by using Eqs. (3.1-5) and (3.1-7). By differentiating Eq. (3.1-5) with respect to \( t \), we obtain

\[ \dot{x}(t) = \dot{x}(t) + \epsilon \dot{\eta}(t) \]  

(3.1-8)
These relations follow as a consequence of the following lemma.

If \( x(t) \) is continuous on the closed interval \( t \in [t_1, t_2] \) and if \( \int_{t_1}^{t_2} x(t) \eta(t) \, dt = 0 \) for every \( \eta(t) \) contained in \( [t_1, t_2] \) such that \( \eta(t_1) = \eta(t_2) = 0 \), then \( x(t) = 0 \) for all \( t \) in \( [t_1, t_2] \). Proof of this lemma is given in reference [1].

These two very important relationships form a good foundation for solving variational problems. Equation (3.1-12) is commonly known as the Euler-Lagrange equation and Eq. (3.1-13) is the associated transversality condition. These equations specify a two-point boundary value differential equation which, when solved, determines \( \phi \) in terms of a known \( \phi_0 \).

3.2 Remarks on transversality conditions.

The various forms and uses of the transversality conditions will be covered in some detail in this chapter. We do this because these conditions are among the hardest things to correctly formulate for any variational problem, and they are generally different enough for each problem to warrant comment.

We will now examine Eq. (3.1-13) and tabulate many of the possible combinations for which this equation holds. In each case, \( t_0 \) and \( t_f \) are fixed.

I. Fixed Beginning-Terminal Points
In this case we fix \( x(t_0) \) and \( x(t_f) \). Thus every admissible solution must pass through these fixed points. Therefore from Eq. (3.1-4) we see that we must require that \( \eta(t_0) = \eta(t_f) = 0 \). In this case the correct boundary conditions are the specified \( x(t_0) \) and \( x(t_f) \).

II. Variable Beginning-Terminal Points
We now consider that \( x(t_0) \) and \( x(t_f) \) are variable or, in other words, not constrained. Therefore from Eq. (3.1-13) we have (since \( \eta(t) \) can be arbitrary at the end points) \( \partial \phi / \partial \dot{x} = 0 \) at \( t = t_0 \) and \( t = t_f \). When this particular situation results, the boundary conditions are called the natural boundary conditions.

III. Variable Beginning-Fixed Terminal Points
In the case where \( x(t_0) \) is variable and \( x(t_f) \) is fixed, we must constrain \( \eta(t_f) \) to be zero but can allow any (admissible) \( \eta(t_0) \). Therefore from Eq. (3.1-13) we have the two-point boundary conditions \( \partial \phi / \partial \dot{x} = 0 \) at \( t = t_0 \), and \( \eta(t_f) = 0 \), which means that the other boundary condition is the specified \( x(t_f) \).

IV. Fixed Beginning-Variable Terminal Points
For \( x(t_0) \) fixed and \( x(t_f) \) variable, a situation which often occurs in optimal control, we have from Eq. (3.1-13) that (since \( \eta(t_f) \) is arbitrary) the two-point boundary conditions are the specified \( x(t_0) \) and \( \partial \phi / \partial \dot{x} = 0 \) at \( t = t_f \).

With this tabulation, the analysis of the scalar Lagrange problem (which, as previously mentioned, includes the scalar Bolza problem's nearly complete. Figure 3.2-1 illustrates graphically the essence of this tabulation.

3.3 The second variation: sufficient conditions for (weak) extrema

Until now, in the study of extrema of functionals we have only considered a necessary condition for a functional to have a relative or weak extremum. This was, of course, the condition that the first variation vanish. In this section, we shall be briefly concerned with sufficient conditions for a function to have extrema and shall thus introduce the second variation. The next section on examples will illustrate the application of the second variation in a particularly simple case.

To establish the nature of an extremum, it is necessary to obtain \( \partial^2 J / \partial \epsilon^2 \) evaluated at \( \epsilon = 0 \) from Eq. (3.1-1) under the conditions of Eq. (3.1-5). This is
Applying integration by parts and the transversality conditions [Eq. (3.1-13)] we have
\[ 2 \int_0^t \eta \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \, \eta \, dt = - \int_0^t \left( \frac{d}{dt} \frac{\partial \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{\theta}} \right) \eta \, dt \]
(3.3-2)
Thus the second variation of \( J \) becomes
\[ \frac{\partial^2 J (x)}{\partial \eta \partial \hat{\theta}} \bigg|_{\eta = \hat{\theta} = 0} = \int_0^t \left[ \eta \left( \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \right) + \eta \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \right] \, dt \]
(3.3-3)
To establish a minimum (maximum) of \( J \), the first necessary condition is that \( \partial J / \partial \eta = 0 \) at \( \eta = 0 \) independently of the variation \( \eta (t) \). The second necessary condition for a minimum (maximum) is that the second derivative of \( J \) with respect to \( \eta \), evaluated at \( \eta = 0 \), be equal to or greater than (equal to or less than) zero. Sufficient conditions for a weak minimum (maximum) require that the derivative be positive (negative). All of this must, of course, be true independent of the variation \( \eta (t) \) and need only be true along the optimal "trajectory," \( \hat{x} (t) \).

We can rewrite Eq. (3.3-1) as the quadratic form integral
\[ \frac{\partial^2 J (x)}{\partial \eta \partial \hat{\theta}} \bigg|_{\eta = \hat{\theta} = 0} = \int_0^t \left[ \eta \left( \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \right) + \eta \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \right] \, dt \]
(3.3-4)
If the matrix in this expression is at least positive (negative) semidefinite, we have certainly established a minimum (maximum). Alternately, from Eq. (3.3-3) we are assured that the second derivative is equal to or greater than zero if
\[ \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \bigg|_{\eta = \hat{\theta} = 0} \geq 0 \]
(3.3-5)
and
\[ \frac{\partial^2 \phi (\hat{x}, \hat{\theta}, t)}{\partial \hat{x} \partial \hat{\theta}} \bigg|_{\eta = \hat{\theta} = 0} \geq 0 \]
(3.3-6)
For many problems in which we will have interest, the foregoing conditions are fulfilled, and we can establish necessary and sufficient conditions for a minimum. It is still possible, however, for Eq. (3.3-1) or Eq. (3.3-2) to be greater than zero even if the requirements of Eqs. (3.3-4), (3.3-5), and (3.3-6) are not satisfied, since \( \eta (t) \) and \( \dot{\eta} (t) \) are not independent of one another.

Complete exploitation of this point is beyond the intent of this chapter. Chapters 5 and 6 of reference [1] provide an excellent and readable discussion of the necessary and sufficient conditions for a minimum. We will return again to this point in Chapter 4. We must again emphasize here that we are establishing conditions for a relative extremum, sometimes called a weak extremum, which may or may not be an absolute extremum. In Section 4.1 we will discuss some requirements for an absolute or strong extremum.

**Example 3.3-1**

We desire to find the curve with minimum arc length between the point \( x (0) = 1 \) and the line \( x = 2 \).

The first step toward solving this problem is to formulate the functional \( J (x) \). If we define the differential arc length as \( ds \), the functional we desire to minimize is easily seen to be
\[ J (x) = \int_0^t ds \]
with associated boundary conditions
\[ x (t = 0) = 1, \quad x (t = 2) = \text{open} \]
Noting that for a differential arc length
\[ (ds)^2 = (dx)^2 + (dt)^2 \]
we have
\[ ds = \sqrt{1 + x^2} \, dt \]
By substituting into the given cost function, we obtain
\[ J (x) = \int_0^t \sqrt{1 + x^2} \, dt \]
Upon referring back to the functional defined in Eq. (3.1-1), we see that
\[ \phi (x, \dot{x}, t) = [1 + \dot{x}^2]^{1/2} \]
The Euler-Lagrange equation for this problem is therefore
\[ \frac{\partial \phi}{\partial \dot{x}} - \frac{d}{dt} \left( \frac{\partial \phi}{\partial \dot{x}} \right) = 0 \]
and thus we obtain
\[ -\frac{d}{dt} \left( \frac{\dot{x}}{1 + \dot{x}^2} \right) = 0 \]
Upon integrating, we obtain
\[ \frac{\dot{x}}{1 + \dot{x}^2} = \text{constant}, \quad \dot{x} = c \frac{1}{\sqrt{1 - c^2}} = a^2 \]
Thus we see that the extremal curve is given by
\[ \hat{x} (t) = at + b \]
Therefore, the shortest distance between a point and a straight line is another straight line.

We obtain the particular solution by properly applying the transversality equation to the given boundary conditions. We note that this problem falls into situation IV, i.e., fixed beginning—variable terminal point. Thus, \( x(t) = x(0) = 1 \) and

\[
\frac{\partial \phi}{\partial \dot{x}} = 0 = \frac{\dot{x}}{(1 + \dot{x}^2)^{3/2}} \quad \text{at} \quad t = 2
\]

or \( \dot{x} = 0 \) at \( t = 2 \).

Differentiating the solution for \( \dot{x} \) with respect to \( t \), we have \( \ddot{x} = a \), and using the transversality conditions we obtain \( a = 0 \) and \( b = 1 \). Therefore, the extremal curve satisfying the given boundary condition and minimizing the given arc length is \( x = 1 \).

To mathematically demonstrate that we have obtained a minimum rather than a maximum or stationary point, it is necessary to show that the second variation is greater than zero, and we have indeed established a minimum. Physically this was, of course, evident from the start.

**Example 3.3-2**

We desire to find the equation of the curve which minimizes the functional (boundary conditions unspecified)

\[
J(x) = \int_{0}^{t} \left[ k \dot{x}^2 + x \ddot{x} + \ddot{x} + x \right] dt
\]

The Euler-Lagrange equation for this problem is

\[
\ddot{x} + 1 - \dot{x} - x = 0 = 1 - \dot{x}
\]

By integrating directly, we obtain the solution to this equation:

\[
x(t) = \frac{t^2}{2} + C_1 t + C_2
\]

To determine \( C_1 \) and \( C_2 \) we must now apply the transversality equation to the given boundary conditions. Since this is a variable beginning—terminal point problem, situation II is used, which is the natural boundary condition case.

\[
\frac{\partial \phi}{\partial \dot{x}} = \dot{x} + x + 1 = 0 \quad \text{for} \quad t = 0, 2
\]

Therefore, from the solution for \( x \) and its derivative, we have

\[
\frac{\partial \phi}{\partial \dot{x}} = t + C_1 + \frac{t^2}{2} + C_2 t + C_2 + 1 = 0 \quad \text{for} \quad t = 0, 2
\]

We can now solve for \( C_1 \) and \( C_2 \) from the simultaneous equations

\[
C_1 + C_2 = -1, \quad 3C_1 + C_2 = -5
\]

to obtain \( C_1 = -2 \) and \( C_2 = 1 \). Therefore the extremal curve which satisfies the given boundary conditions, is

\[
x(t) = \frac{t^2}{2} - 2t + 1
\]

The actual value of the extremum is obtained when we substitute into the given cost function and carry out the integration to obtain \( J_{\text{min}} = \frac{\pi}{4} \).

### 3.4 Unspecified terminal time problems

By slightly changing the cost function given in Eq. (3.1-1) we obtain a very useful problem formulation; it is called an unspecified terminal time problem and, as will be apparent later, leads to the "minimum time" problem of optimal control. The basic problem is one of minimizing a given cost function where \( t_f \) is unspecified subject to the constraint that the final state of the system be specified by a prescribed terminal line or, in higher-dimension problems, terminal manifold.

The cost function generally contains terms representing energy expended, distance traversed, elapsed time, and so forth, which may appear singly or in combination. The original state of the system may be specified or unspecified, and the terminal line or manifold may be time-varying or invariant.

The approach used here will be general enough so that any of the foregoing specifications can be included in the solution of a specific problem. A graphical illustration of a variable terminal time problem is given in Fig. 3.4-1. Instead of calling \( J(x) \) a functional, we will now use the systems control terminology, cost function, which for this problem will be given by

\[
J(x) = \int_{t_0}^{t_f} \phi(x, \dot{x}, t) \, dt \quad (3.4-1)
\]

where \( t_0 \) is known, \( t_f \) is unspecified, and \( x(t_0) \) may or may not be specified.

---

**Fig. 3.4-1.** Illustration of variable terminal time problem where \( x(t_f) = c(t_f) \).
We should note that for the problem shown in Fig. 3.4-1 the initial state, \(x(t_0)\), is specified though, in general, as previously stated, it need not be.

As before, \(\dot{x}(t)\) is the required curve, here referred to as the optimal system trajectory. A family of curves, which includes the optimal trajectory \(\dot{x}(t)\), starting at \(t_0\) and ending at \(t_f\) is given by

\[
x(t) = \dot{x}(t) + \eta(t)\]

with time derivative

\[
\dot{x}(t) = \dot{\dot{x}}(t) + \dot{\eta}(t)
\]

where \(\eta(t)\) is a variation in \(x\) which depends on \(t\).

Since the terminal time is unspecified, it must be treated as a variable and, therefore, must be examined to see if perhaps there is a final time, \(t_f\), which is optimal. We will therefore define a family of final times, one of which is the optimal final time \(t_f\):

\[
t_f = t_f + \eta(t_f)
\]

where \(\eta(t_f)\) is a variation in \(t_f\).

Our first step in minimizing the cost function, Eq. (3.4-1), is to substitute Eqs. (3.4-2), (3.4-3), and (3.4-4) into it, which gives us

\[
J(x) = \int_{t_0}^{t_f + \eta(t_f)} \phi(\dot{x}(t) + \dot{\eta}(t), \dot{\eta}(t), \eta) dt
\]

We now set \(\delta J/\delta \eta = 0\) at \(e = 0\) and obtain

\[
\begin{align*}
\frac{\delta J}{\delta \eta} |_{e=0} = 0 &= \int_{t_0}^{t_f} \left[ \eta(t) \frac{\delta \phi}{\delta \dot{x}} + \dot{\eta}(t) \frac{\delta \phi}{\delta \dot{\eta}} \right] dt + \eta(t_f) \phi(\dot{x}(t_f), \dot{\eta}(t_f), \eta(t_f)) \\
&= 0
\end{align*}
\]

Integrating a portion of Eq. (3.4-6) by parts, we obtain

\[
\int_{t_0}^{t_f} \eta(t) \left[ \frac{\delta \phi}{\delta \dot{x}} - \frac{d}{dt} \frac{\delta \phi}{\delta \dot{x}} \right] dt + \eta(t_f) \phi(\dot{x}(t_f), \dot{\eta}(t_f), \eta(t_f)) = 0
\]

At the terminal time, the terminal line, \(C(t_f)\), or, in higher dimensions, \(\dot{x}\) terminal manifold, and the optimal trajectory \(\dot{x}(t)\) intersect, as shown in Fig. 3.4-1. Therefore, using Eqs. (3.4-2) and (3.4-4), we have

\[
\dot{x}(t_f + \eta(t_f)) + \eta(t_f) \dot{x}(t_f) = C(t_f + \eta(t_f))
\]

We take the partial derivative of this equation with respect to \(e\) and evaluate it at \(e = 0\) to obtain

\[
\eta(t_f) \dot{x}(t_f) + \dot{\eta}(t_f) = \eta(t_f) \dot{C}(t_f)
\]

where \(\dot{x}(t) = \delta x/\delta t\) and \(\dot{C}(t) = \delta C/\delta t\) at \(t = t_f\). Thus

\[
\eta(t_f) = \eta(t_f) \dot{C}(t_f) - \dot{x}(t_f)
\]

By substituting Eq. (3.4-10) into (3.4-7), we have

\[
\int_{t_0}^{t_f} \eta(t) \left[ \frac{\delta \phi}{\delta \dot{x}} - \frac{d}{dt} \frac{\delta \phi}{\delta \dot{x}} \right] dt + \eta(t_f) \phi(\dot{x}(t_f) - \dot{x}(t_f)) + \phi(\dot{x}(t_f), \dot{\eta}(t_f), \eta(t_f)) - \eta(t_f) \frac{\delta \phi}{\delta \eta} |_{e=0} = 0
\]

Remembering that Eq. (3.4-11) must be identically equal to zero independent of the variations, we see that the first requirement for the solution to our problem (the second variation must also be non-positive) is that

\[
\eta(t_f) \frac{\delta \phi}{\delta \eta} |_{e=0} = 0, \quad \text{for } t = t_f
\]

\[
\eta(t) \frac{\delta \phi}{\delta \eta} |_{e=0} = 0, \quad \text{for } t = t_f
\]

We recognize that Eq. (3.4-12) is the familiar Euler-Lagrange equation while Eqs. (3.4-13) and (3.4-14) comprise the transversality conditions for this problem. As before, there are four different relationships obtainable from the transversality conditions, but since they are so similar to those discussed previously, the details of these relationships are left as an exercise. We note that the \(\phi\) notation has been removed from Eqs. (3.4-12) through (3.4-14) for convenience. Let us now attempt to apply our results to a simple problem.

**Example 3.4-1**

We wish to minimize

\[
J(x) = \int_{t_0}^{t_f} [1 + x^2] \, dt
\]

with \(x(0) = 1\) such that \(x(t) - C(t) = 2 - t_f\).

We should recognize that the cost function is actually the arc length, which means that the distance between a point and a line is being minimized. Application of the Euler-Lagrange equation yields the optimal trajectory \(x = at + b\), as in Example 3.3-1. To evaluate the arbitrary constants \(a\) and \(b\), we make proper use of the transversality Eqs. (3.4-13) and (3.4-14). Here we specify \(x(0) = 1\); thus \(\eta(0) = 0\). And since \(t_f\) is unspecified, Eq. (3.4-13) becomes

\[
(C - \dot{x}) \frac{\delta \phi}{\delta \eta} + \dot{\phi} = 0, \quad \text{at } t = t_f
\]

Thus we obtain \(a = 1\) at the unspecified terminal time \(t_f\). From the solution to the Euler-Lagrange equation and the specified initial condition, we have \(x(t = 0) = 1\); so we must have \(b = 1\) and \(x(t = t_f) = a = 1\). Therefore the optimal trajectory is \(x(t) = t + 1\), and the final time \(t_f\) is \(t_f = 1\). Salient features of this problem are indicated in Fig. 3.4-2. An interesting fact here is that the optimal trajectory intersects the terminal manifold at right angles. In general,
the optimal trajectory will always be nontangent to the terminal manifold. This nontangency condition is, in fact, called the transversality condition.

3.5 Euler-Lagrange equations and transversality conditions—vector formulation

The previous results can be easily generalized to include scalar cost functions in n-dimensional variables via the state-space approach. That is, we desire to minimize

$$J = \int_{t_0}^{t_1} \phi(x, \dot{x}, t) \, dt$$

where $x$ is the system state, an $n$ vector such that $x^T = [x_1, x_2, \ldots, x_n]$. $t_0$, the starting time, is generally specified (it may not be); $x(t_0)$ may or may not be specified; $x(t_f)$ is specified by a given terminal manifold denoted $C(t_f)$. As before, the terminal time $t_f$ does not have to be known. After following a procedure quite similar to the scalar one, we have, after setting $\delta J/\delta \epsilon = 0$ and dropping the $^\epsilon$ notation the requirement that among other things

$$\int_{t_0}^{t_f} \eta(t) \left[ \frac{\partial \phi}{\partial x} - \frac{d}{dt} \frac{\partial \phi}{\partial \dot{x}} \right] \, dt = 0 \quad (3.5-2)$$

be true independent of $\eta(t)$. This leads to the requirement that

$$\frac{\partial \phi}{\partial x} - \frac{d}{dt} \frac{\partial \phi}{\partial \dot{x}} = 0 \quad (3.5-3)$$

Although it may seem that all unspecified terminal time problems may now be worked by mere substitution into the derived relationships, Eqs. (3.5-3) through (3.5-6), this is not the case. Many problems do not fall precisely into a form which allows direct use of our derived formulas. When this type of problem is encountered, a good procedure to follow is to derive the transversality condition for the particular problem. An example demonstrating this type of approach follows.

Example 3.5-2

We wish to find the transversality conditions for the minimization of

$$J = \int_{t_0}^{t_f} \phi(x, \dot{x}, t) \, dt$$

such that $\|x(t_f)\| = 1$, where $x^r = [x_1, x_2]$, with specified starting time $t_0$ and terminal time $t_f$. Thus, we would like to reach the region of state-space specified...
by $x_1^2 + x_2^2 = 1$ specified terminal time $t_f$ given the state at the starting time
$t_0$, denoted by $x(t_0)$.

The transversality conditions are, from Eq. (3.5-4),

$$
\frac{\partial f}{\partial x_1} \eta_{x_1} + \frac{\partial f}{\partial x_2} \eta_{x_2} = 0, \quad \text{at} \quad t = t_f
$$

As before, we assume that $x(t) = \dot{x}(t) + \eta_x(t)$ where $x$ is the optimal trajectory. For this problem, this relation in component form becomes $x_1 = \dot{x}_1 + \eta_{x_1}$ and $x_2 = \dot{x}_2 + \eta_{x_2}$. Substituting these results into the given terminal manifold, we obtain

$$(\dot{x}_1 + \eta_{x_1})^2 + (\dot{x}_2 + \eta_{x_2})^2 = 1, \quad \text{at} \quad t = t_f$$

Taking the partial derivative of the foregoing equation with respect to $\epsilon$ and then setting $\epsilon = 0$, we have

$$\dot{x}_1 \eta_{x_1} + \dot{x}_2 \eta_{x_2} = 0, \quad t = t_f$$

We thus see that the specification of the terminal manifold

$$x(t_f) + x(\dot{x}(t_f)) = 1$$

leads to a linear relationship between $\eta_{x_1}$ and $\eta_{x_2}$ at the terminal time. If we combine this relation with the previously stated transversality condition, we obtain for one of the terminal boundary conditions

$$\frac{\partial \phi}{\partial x_1} x_1 + \frac{\partial \phi}{\partial x_2} x_2 = 0, \quad \text{at} \quad t = t_f$$

Therefore the two boundary conditions at $t = t_f$ are

$$x(t_f) + x(\dot{x}(t_f)) = 1$$

$$\frac{\partial \phi}{\partial x_1} x_1(t_f) + \frac{\partial \phi}{\partial x_2} x_2(t_f) = 0$$

Thus for a given $\phi(x, \dot{x}, t)$, we can resolve this problem completely by solving

for the optimal trajectory through the Euler-Lagrange equations and the appropriate boundary conditions which we have just obtained.

### 3.6 Variational notation

Much of the notation in the problems that follow can be considerably simplified if variational rather than differential notation is used. We wish to minimize (for $t_0$ and $t_f$ fixed)

$$J = \int_{t_0}^{t_f} \phi(x, \dot{x}, t) \, dt \quad (3.6-1)$$

We assume, as in Section 3.1, that both $x(t)$ and $\dot{x}(t)$ are representable by a family of curves

$$x(t) = \hat{x}(t) + \eta(t), \quad \dot{x}(t) = \dot{\hat{x}}(t) + \eta(t) \quad (3.6-2)$$

where $x(t)$ is the optimal (extremal) curve and $\eta(t)$ is a variation in $x(t)$ depending upon $t$. We substitute Eq. (3.6-2) into Eq. (3.6-1). id expand $\phi(x, \dot{x}, t)$ in a Taylor series about the point $\xi = 0$.

$$\phi[\hat{x}(t) + \eta(t), \dot{x}(t) + \eta(t), t] = \phi(\hat{x}(t), \dot{x}(t)) + \frac{\partial \phi}{\partial x} \eta(t) + \frac{\partial \phi}{\partial \dot{x}} \dot{\eta}(t) + \text{H.O.T.} \quad (3.6-3)$$

where H.O.T. is used to indicate higher-order terms in $\eta(t)$ and $\dot{\eta}(t)$.

If we now let

$$\Delta J = J(\hat{x} + \eta) - J(\hat{x})$$

we can write

$$\Delta J = \left[ \int_{t_0}^{t_f} \left( \frac{\partial \phi}{\partial x} \eta + \frac{\partial \phi}{\partial \dot{x}} \dot{\eta} \right) + \text{H.O.T.} \right] \, dt$$

Now we define the first variation of $x(t)$ and $\dot{x}(t)$ as

$$\eta(t) = \delta x, \quad \dot{\eta}(t) = \delta \dot{x} \quad (3.6-5)$$

Thus,

$$\Delta J = \left[ \int_{t_0}^{t_f} \left( \frac{\partial \phi}{\partial x} \delta x + \frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} + \text{H.O.T.} \right) \right] \, dt$$

Since the variation plays the same role in variational calculus as the differential in standard calculus, we use the property of linearity, which means that

$$\delta J = \left[ \int_{t_0}^{t_f} \left( \frac{\partial \phi}{\partial x} \delta x + \frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} \right) \right] \, dt$$

(3.6-7) A necessary condition for an extremum at $x(t) = \hat{x}(t)$, i.e., $\epsilon = 0$, is that the first variation of $J, \delta J$, be zero. Applying this to Eq. (3.6-7), along with the minor simplification of integrating by parts and dropping the $\sim$ notation, we obtain

$$\int_{t_0}^{t_f} \frac{\partial \phi}{\partial x} \delta x \, dt + \frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} \bigg|_{t_0}^{t_f} = 0$$

(3.6-8) For Eq. (3.6-8) to equal zero independent of the variation $\delta x$, we must have

$$\frac{\partial \phi}{\partial x} \frac{d}{dt} \delta x + \frac{\partial \phi}{\partial \dot{x}} \delta \dot{x} = 0$$

(3.6-9) $\frac{\partial \phi}{\partial x} \delta x = 0$, for $t = t_0, t_f$ (3.6-10)

We note that Eq. (3.6-9) is the Euler-Lagrange equation and Eq. (3.6-10) is its associated transversality condition.
In a similar manner, it is also easy to show that the second variation of Eq. (3.6-1), written \( \delta^2 J \), is

\[
\delta^2 J = \frac{1}{2} \int_{a}^{b} \left( \delta x \right)^T \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{d}{dt} \frac{\partial^2 \phi}{\partial x \partial t} \right) \left( \delta x \right) dt
\]

where the second variation is now defined as the quadratic part of Eq. (3.6-6) or twice Eq. (3.3-4). As previously stated, the interpretations of the second variation are that \( \delta^2 J > 0 \) implies a minimum of \( J \) and \( \delta^2 J \approx 0 \) implies a maximum of \( J \). A quadratic form integral similar to Eq. (3.3-4) also follows directly.

3.7 Dynamic optimization with equality constraints—Lagrange multipliers

A constrained optimization problem may require extremizing a cost function of the form

\[
J = \int_{a}^{b} \phi(x, \dot{x}, t) dt
\]

subject to the equality constraint

\[
\Lambda(x, \dot{x}, t) = 0
\]

where \( x^T = [x_1, x_2, \ldots, x_n] \) and \( \Lambda^T = [\Lambda_1, \Lambda_2, \ldots, \Lambda_m] \) with \( m \leq n \). It can be shown that the solution to this problem is the same as that obtained by extremizing

\[
J' = \int_{a}^{b} [\phi(x, \dot{x}, t) + \dot{\Lambda}^T(t) \Lambda(x, \dot{x}, t)] dt
\]

where \( \Lambda^T = [\lambda_1, \lambda_2, \ldots, \lambda_m] \) is the vector equivalent of the Lagrange multiplier discussed in Chapter 2 [4].

To illustrate the development of the Lagrange multiplier, let us consider a special case where \( x \) is a two vector. Suppose that we wish to minimize

\[
J = \int_{a}^{b} \phi(x_1, x_2, t) dt
\]

subject to the constraint (with fixed end points)

\[
\Lambda(x_1, x_2, t) = 0
\]

We will use the variational notation just developed to establish a method for treating the given equality constraint. To establish a minimum, it is necessary that the first variation of Eq. (3.7-4) be zero, that is

\[
\delta J = \int_{a}^{b} \left( \delta x_1 \left[ \frac{\partial \phi}{\partial x_1} - \frac{d}{dt} \frac{\partial \phi}{\partial x_1} \right] + \delta x_2 \left[ \frac{\partial \phi}{\partial x_2} - \frac{d}{dt} \frac{\partial \phi}{\partial x_2} \right] \right) dt = 0 \tag{3.7-6}
\]

If \( \delta x_1 \) were independent of \( \delta x_2 \), we could simply set each term of Eq. (3.7-6) equal to 0. Since the constraint provides a dependence on \( x_1 \) and \( x_2 \), we must take the given constraint into consideration. Taking the variation of Eq. (3.7-5) we have

\[
\delta \Lambda = \frac{\partial \Lambda}{\partial x_1} \delta x_1 + \frac{\partial \Lambda}{\partial x_2} \delta x_2 = 0 \tag{3.7-7}
\]

It also follows that, for any \( \lambda(t) \), we may multiply Eq. (3.7-7) by \( \lambda(t) \) and integrate so that

\[
\int_{a}^{b} \lambda(t) \left[ \frac{\partial \Lambda}{\partial x_1} \delta x_1 + \frac{\partial \Lambda}{\partial x_2} \delta x_2 \right] dt = 0 \tag{3.7-8}
\]

If we add Eq. (3.7-6) to Eq. (3.7-8) we obtain

\[
0 = \int_{a}^{b} \left[ \delta x_1 \left[ \frac{\partial \phi}{\partial x_1} - \frac{d}{dt} \frac{\partial \phi}{\partial x_1} + \lambda \frac{\partial \Lambda}{\partial x_1} + 2 \delta x_2 \left[ \frac{\partial \phi}{\partial x_2} - \frac{d}{dt} \frac{\partial \phi}{\partial x_2} + \lambda \frac{\partial \Lambda}{\partial x_2} \right] \right] dt \right.

(3.7-9)

We will now adjust \( \lambda \) so that the term within the first brackets under the integral is zero. It also must follow that, since \( \delta x_1 \) is arbitrary, the term in the second brackets under the integral is also equal to zero. It is apparent that we would have obtained the same results had we reformulated the given problem by adjoining to the cost function the constraint via a Lagrange multiplier as in Eq. (3.7-3) and used the Euler-Lagrange equations on this cost function. The resulting Euler-Lagrange equations would then be solved subject to the equality constraint of Eq. (3.7-2).

Example 3.7-1

We are given the differential system

\[
\dot{\theta} = u(t)
\]

which may be interpreted as the moment of inertia of a rocket in free space, and we desire to minimize

\[
J = \frac{1}{2} \int_{0}^{\theta} (\dot{\theta})^2 dt
\]

such that

\[
\theta(t = 0) = 1, \quad \theta(t = 2) = 0
\]

To cast this problem in state space notation, we let

\[
x_1(t) = \theta(t), \quad x_1 = x_2(t), \quad x_2 = u(t)
\]

Now the differential system can be represented by

\[
\dot{x} = Ax(t) + bu(t)
\]

where

\[
x^T = [x_1, x_2], \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b^T = [0, 1] \]
When we apply (3.7-3) \( u(t) \) is treated as another state variable, \( x_3 \), the problem becomes one of minimizing

\[
J = \int_0^T \left( \frac{1}{2} u^2(t) + \lambda_1(t) [x_3(t) - x_1] + \lambda_2(t) [u(t) - \dot{x}_2] \right) dt
\]

The Euler-Lagrange equations yield

\[
\dot{x}_1 = 0, \quad \dot{x}_2 = -\lambda_2(t), \quad u(t) = -\lambda_1(t)
\]

The final solution is obtained by means of the given differential relationships and boundary conditions, and it is

\[
x_1 = \frac{1}{2} t^2 - \frac{1}{2} t^2 + t + 1, \quad x_2 = \frac{3}{2} t^2 - \frac{3}{2} t + 1, \quad u = 3t - \frac{1}{2}
\]

This system, along with a plot of the system trajectories, is shown in Fig. 3.7-1.

---

Example 3.7-2 Linear Servomechanism

Suppose that we wish to minimize

\[
J = \frac{1}{2} \int_{t_0}^T \|u(t)\|_R(t) + \|x(t) - r(t)\|_Q(t) dt
\]

for the general time-varying system specified by

\[
\dot{x} = A(t)x(t) + B(t)u(t)
\]

with \( x(t_0) = x_0 \) as the initial condition vector, \( r(t) \) is the desired value of the state vector \( x(t) \). As before, it is necessary to assume that all matrices and vectors are of compatible orders. We adjoin the differential system equality constraint to the cost function by the Lagrange multiplier to obtain

\[
J' = \frac{1}{2} \int_{t_0}^T \left[ \frac{1}{2} u^2(t) + \|x(t) - r(t)\|_Q(t) + \lambda^T(t)x(t) + B(t)u(t) - \dot{x} \right] dt
\]

The exact nature of the cost function used depends upon the particular problem being solved. Therefore \( R(t) \) and \( Q(t) \), both penalty-weighting matrices, are generally chosen with regard to the physical conditions present. We also assume that both \( R(t) \) and \( Q(t) \) are symmetric, since there is no loss in generality by doing so. The control vector, \( u(t) \) is treated just as if it were a state vector. Then we apply the Euler-Lagrange equations, which in this case are

\[
\frac{\partial \Phi}{\partial x} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{x}} = 0, \quad \frac{\partial \Phi}{\partial u} - \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{u}} = 0
\]

where

\[
\Phi = \frac{1}{2} u(t) R(t) u(t) + \|x(t) - r(t)\|_Q(t) + \lambda^T(t) [A(t)x(t) + B(t)u(t) - x]
\]

Thus

\[
\frac{\partial \Phi}{\partial x} = Q(t) [x(t) - r(t)] + \lambda^T(t) A(t), \quad \frac{\partial \Phi}{\partial u} = -\lambda(t)
\]

\[
\frac{\partial \Phi}{\partial u} = R(t) u(t) + B^T(t) \lambda(t), \quad \frac{\partial \Phi}{\partial \dot{u}} = 0
\]

The Euler-Lagrange equations for this problem become

\[
\dot{x} = -A^T(t) \lambda(t) - Q(t) [x(t) - r(t)], \quad u(t) = -R^{-1}(t) B^T(t) \lambda(t)
\]

Since \( x(t_f) \) is unspecified, the transversality condition at the terminal time yields \( \lambda(t_f) = 0 \). This solution can be block-diagrammed as in Fig. 3.7-2. We note that the solution for the optimal control requires that \( R(t) \) have an inverse. Also, certain other requirements must be met to insure a minimum of the cost function; specifically, \( R(t) \) and \( Q(t) \) must be nonnegative definite to insure a nonnegative second variation. Thus we see that \( R(t) \) must be positive definite.

Although it appears that we have solved the originally stated problem, there are still some further refinements which are highly desired. Since the state of the system is specified at \( t_0 \), we are given \( x(t_0) \), while the adjoint operator \( \lambda(t) \) is specified at the terminal time, \( \lambda(t_f) = 0 \). What we, in fact, have to do is solve a two-point boundary value problem (TPBVP), something which, in general, cannot always be done without recourse to electronic computers. In this particular case, since the differential equations are all linear, superposition can be invoked and a closed-form analytical solution obtained with great difficulty.

If we let \( r(t) \) be either a constant vector or the null vector, the foregoing problem reduces to a regulator problem. The treatment of the servomechanism problem can be made more general if we assume that indirect state observation is made available to us, that is, for the system

\[
\dot{x} = A(t)x(t) + B(t)u(t)
\]
Fig. 3.7-2. Block diagram of a possible solution to the servomechanism problem.

we can obtain directly only

\[ z(t) = C(t)x(t) + D(t)u(t) \]

The procedure and results are quite similar to the ones obtained in this example except that requirements on observability and controllability, to be discussed in Chapter 11, are present.

To solve this two-point boundary value problem, we must require a knowledge of \( r(t) \) for all time in the closed interval \( t_0 \) to \( t_1 \) or, in shorthand notation, \( VIE_{[t_0, t_1]} \). Since a two-point boundary value problem must be solved before we can determine the optimum control for this problem, it is clear that a closed-loop control has not been found. After we have formulated the Hamilton-Jacobi equations and the Pontryagin maximum principle, we will have a great deal more to say about this important problem.

### 3.8 Dynamic optimization with inequality constraints

In many physical problems of interest to the control engineer, there are various inequality constraints on the control vector. For example, the maximum thrust from a reaction jet is physically limited as is the maximum input reactivity in a nuclear reactor. When inequality constraints are present, it is necessary that we consider them in determining optimum system design.

Thus we are faced with minimizing a cost function of the form

\[ J = \int_{t_0}^{t_f} \phi(x, \dot{x}, t) \, dt \]  

with equality constraints of the form

\[ \Lambda(x, \dot{x}, t) = 0 \]

and inequality constraints of the form

\[ \Gamma_{\text{min}} \leq \Gamma(x, \dot{x}, t) \leq \Gamma_{\text{max}} \]

When the inequality constraint involves the control vector, the control vector which satisfies the constraint conditions is called an admissible control vector. One technique which is generally satisfactory for resolving the control inequality constraint problem consists of converting the inequality constraint to an equality constraint. It can be easily demonstrated that the equations

\[ (\Gamma_{\text{max}} - \Gamma_i)(\Gamma_1 - \Gamma_{\text{min}}) = \eta_i, \quad i = 1, 2, \ldots \]  

are equivalent to the constraints of Eq. (3.8-3), since each term on the left side of Eq. (3.8-4) must be positive and thus have a positive product. Thus the inequality constraints have been converted to equality constraints and may be treated as such. Lagrange multipliers are then used to adjoin the equality and inequality constraints to the cost function, Eq. (3.8-1), and the Euler-Lagrange equations applied. The technique can best be illustrated by an example.

**Example 3.8-1**

Let us consider the same plant dynamics as in the previous example

\[ \dot{x}_1 = x_2(t), \quad \dot{x}_2 = u(t) \]

with the initial conditions \( x_1(t_0) = x_a \) and \( x_2(t_0) = v_0 \). The problem is to find the control which maximizes \( x_1(t_f) \) for fixed \( t_f > t_0 \) subject to the boundary condition equality constraint that \( x_1(t_f) = v_f \) and the inequality constraint on the scalar control \( u_{\text{min}} \leq u \leq u_{\text{max}} \). We convert the inequality constraint to an equality constraint by introducing a new variable \( \alpha(t) \) and replacing the inequality constraint by

\[ (u - u_{\text{min}})(u_{\text{max}} - u) - \alpha^2 = 0 \]

Thus the problem may be recast as one of minimizing \( J = -x_1(t_f) \) subject to the equality constraints

\[ \dot{x}_1 = x_2(t), \quad x_1(t_0) = x_a, \quad x_1(t_f) = v_f \]

\[ \dot{x}_2 = u(t), \quad x_2(t_0) = v_0, \quad x_2(t_f) = v_f \]

\[ (u - u_{\text{min}})(u_{\text{max}} - u) - \alpha^2 = 0 \]

†Chapters 4, 13, and 14 will consider more varied aspects, theoretical and computational, of the inequality constraint problem.
The cost function with the adjoined Lagrange multiplier becomes
\[ J' = \left[ x_1(t_0) + \int_{t_0}^{t_f} \right] + \lambda_1[x_2 - x_1] + \lambda_2[u - x_1] + \lambda_3[u - u_{\text{min}}(\max) - u_\text{}}dt\right] + \lambda_3[(u - u_{\text{min}})(\max) - u_\text}dt\right] - \chi(x_{1}) \]

The Euler-Lagrange equation
\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} - \frac{\partial \mathcal{L}}{\partial x_1} = 0 \]
with
\[ \Phi = \lambda_1[x_2 - x_1] + \lambda_2[u - x_1] + \lambda_3[(u - u_{\text{min}})(\max) - u_\text}dt\right] - \chi(x_{1}) \]
yields
\[ \dot{x}_1 = 0, \quad \dot{x}_2 = -\lambda_1, \quad 0 = -\lambda_1 + \lambda_3[2u - u_{\text{min}}] \]
\[ \lambda_3 = \alpha \lambda_3 \]
Application of the natural boundary condition equation (transversality condition) to determine the single missing terminal condition on \( x_1(t_f) \) yields
\[ \frac{\partial \Phi}{\partial x_1} |_{x_1(t_f)} = 0 = -1 - \lambda_3(t_f) \]
Thus we have arrived at the two-point boundary value problem whose solution determines the optimal state and control variables. This TPBVP is
\[ \dot{x}_1 = x_2(t), \quad x_1(t_0) = x_0, \]
\[ \dot{x}_2 = u(t), \quad x_2(t_0) = v_0, \]
\[ \dot{x}_1 = 0, \quad \lambda_3(t_f) = -1, \]
\[ \dot{x}_2 = -\lambda_1, \quad x_2(t_f) = v_f, \]
\[ \alpha(t) = \lambda_3(2u(t) - u_{\text{max}} - u_{\text{min}}), \]
\[ \alpha(t) = \lambda_3(u(t) - u_{\text{min}}(\max) - u_\text}dt\right] - \chi(x_{1}) \]
This TPBVP is nonlinear because of the last three coupling equations above and is quite difficult to solve without recourse to a computer. In a usual version of this problem, \( u_{\text{min}} = -1 \) and \( u_{\text{max}} = +1 \). In that case, it is possible to show that \( \alpha(t) = 0 \) and
\[ u(t) = -\text{sign} \lambda_2(t) \]
where
\[ \text{sign} \lambda_2 = 1 \quad \text{if} \quad \lambda_2 > 0 \]
\[ \text{sign} \lambda_2 = -1 \quad \text{if} \quad \lambda_2 < 0 \]
This does not, however, change the nonlinear nature of the two-point boundary problem. In a later chapter we will devote considerable time to various gradient methods, Newton-Raphson techniques, and other computational techniques for solving nonlinear two-point (and multipoint) boundary value problems.

REFERENCES

PROBLEMS

1. A linear differential system is described by
\[ \dot{x} = Ax + Bu \]
where
\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x^0 = [x_1, x_2], \quad u^0 = [u_1, u_2] \]
Find \( u(t) \) such that
\[ J = \frac{1}{2} \int_0^T \|u(t)\|^2 dt \]
is minimum, given \( x^0(0) = [1, 1] \) and \( x(2) = 0 \).
2. Find the conditions necessary for minimizing
\[ J = \int_0^T \phi(x, \dot{x}, t) dt \]
given \( x(t) = x_0 \) and \( g(x, \dot{x}, t) = 0 \).
3. Use the results of Problem 2 to find the control \( u(t) \), which minimizes
\[ J = \int_0^T \frac{x^2(t)}{2} + \frac{1}{2} \int_0^T u^2 dt \]
such that \( \dot{x} = u(t), \quad x(0) = 1 \).
4. A linear system is described by
\[ \dot{x} = -x + u, \quad x(0) = 1 \]
It is desired to minimize
\[ J = \frac{1}{2} \int_0^T (x^2 + u^2) dt \]
A feedback law is obtained if we let \( u(t) = -\alpha x(t) \) where \( d\alpha/dt = 0 \) such that \( \alpha \) is a constant. Find the equations defining the optimum value of \( \alpha \).
5. Find the differential equations and associated boundary conditions whose solutions minimize
\[ J = \frac{1}{2} \int_0^T u^2 dt \]
for the differential system described by
\[ \dot{x}_1 = -x_1 + x_2 \]
\[ \dot{x}_2 = u \]
with end points given by
\[ x_1(0) = x_2(0) = 0 \]
\[ x(T) + x(t_f) = t_f + 1 \]

6. Find the value of \( u \) which minimizes (for \( t_f \) unspecified)
\[ J = \int_0^{t_f} \left[ \alpha + u^2(t) + x^2(t) \right] dt \]
for the differential system
\[ \dot{x} = -x(t) + u(t), \quad x(0) = 1, \quad x(t_f) = 0 \]

7. A linear second-order differential equation is described by
\[ \dot{x}_1 = x_2(t), \quad x_1(0) = 1 \]
\[ \dot{x}_2 = u \quad x_2(0) = 1 \]
Find, by use of the Euler-Lagrange equations and transversality conditions, the optimal control \( u(t) \) which minimizes:
(a) \[ J = \int_0^t u^2 dt, \quad x_1(0) = x_2(0) = 0 \]
(b) \[ J = \int_0^t u^2 dt, \quad x_1(t) = 0 \]
(c) \[ J = \int_0^t u^2 dt, \quad x_1(t) = c(t), \quad x_2(t) = -t^2 \]
(Also determine \( t_f \) and \( x_1(t_f) \).)
(d) \[ J = \int_0^t u^2 dt, \quad x_1(t) = c(t) = -t^2, \quad x_2(t) = 0 \]
(e) \[ J = \int_0^t \left[ \|x\|^2 + \|u\|^2 \right] dt \]
For all cases, sketch both the optimal system trajectory \( x(t) \) and the optimal system control \( u(t) \).

8. For the fixed plant dynamics given by
\[ \dot{x} = u \]
determine the optimal closed-loop system which minimizes
\[ J = \frac{1}{2} \int_0^t [u^2 + (x - \bar{f})^2] \, dt \]
where \( \bar{f}(t) = 1 - e^{-t} \).

9. For the fixed plant dynamics given by \( \dot{x} = u(t), \quad x(0) = x_0 \), determine the optimal closed-loop control which minimizes for fixed \( t_f \)
\[ J = \frac{1}{2} \int_0^{t_f} u^2(t) \, dt + \frac{1}{2} \int_0^{t_f} x^2(t_f) \, dt \]
where \( s \) is an arbitrary constant. Do this by first determining the optimum open-loop control and trajectory and then let \( u(t) = k(t)x(t) \).

In the previous chapter, we formulated many problems in the classical calculus of variations. A derivation of the Euler-Lagrange equations for both the scalar and vector cases was presented. We discussed the associated transversality conditions and some of the difficulties which we may encounter if inequality constraints are present. Several simple optimal control problems were stated and solved. In this chapter we wish to reexamine many of the problems presented in the previous chapter and obtain more general solutions for some of them. In addition, we will develop methods for handling some problems which could not be conveniently formulated by the methods in the previous chapter.

To these ends, we will present the Bolza formulation of the variational calculus using Hamiltonian methods. This will lead us into a proof of the Pontryagin maximum principle and the associated transversality conditions. We will proceed then to a development of the Hamilton-Jacobi equations, which are equivalent to Bellman's equations of continuous dynamic programming. Finally, we will give brief mention to some limitations of dynamic programming. Examples to illustrate the methods will be presented. We will reserve the next chapter for a discussion of some of the many problems which we can formulate and solve using the maximum principle.

In order to fully develop our approach to optimization theory where
4.1 Variation of functions with terminal times 
not fixed—the Weierstrass-Erdmann conditions

In this section, we will consider problems which arise when the terminal (or initial) time is not fixed (unspecified in the problem statement). We must reexamine our concept of a variation in order to accurately treat problems wherein the terminal (or initial) time is not fixed if we are to use the powerful concept of the first variation. We thus wish to consider the extremization of

\[ J = \int_{t_0}^{t_f} \Phi(x(t), x'(t), t) \, dt \]  

(4.1-1)

where all admissible trajectories are smooth and where the terminal time is not fixed. We define a variation \( \delta J \) as the part of

\[ \Delta J = J[x + h, t_r + \delta t_r] - J[x, t_r] \]  

(4.1-2)

which is linear in \( h, \delta x, \delta x', \) and \( \delta t_r \). Since both \( x \) and \( t_r \) vary, it is appropriate to consider the variation \( \delta x \) as

\[ \delta x(t_r) = h(t_r) + \dot{x}(t_r) \delta t_r \]  

(4.1-3)

For the cost function, Eq. (4.1-1), we find that

\[ \Delta J = \int_{t_0}^{t_f} \frac{\partial \Phi}{\partial x} \delta x(t) + \frac{\partial \Phi}{\partial x'} \delta x'(t) + \frac{\partial \Phi}{\partial t} \delta t \]  

(4.1-4)

By taking the linear terms in this equation and performing an integration by parts, we obtain the first variation as:

\[ \delta J = \Phi(x(t_r), x'(t_r), t_r) \delta t_r + \int_{t_0}^{t_f} \frac{\partial \Phi}{\partial t} \delta t_r \]  

(4.1-5)

where, for convenience, we assume that \( h(t_r) = 0 \). Using Eq. (4.1-3), the first variation becomes

\[ \delta J = \left[ \Phi(x(t_r), x'(t_r), t_r) - \dot{x}^2(t_r) \right] \delta t_r \]  

(4.1-6)

\[ + \int_{t_0}^{t_f} \frac{\partial \Phi}{\partial t} \delta t_r \]  

In much of our work, it will be convenient to define a quantity, called the Hamiltonian, by

\[ H[x(t), \lambda(t), t_r] = \Phi - x^2 \frac{\partial \Phi}{\partial x} = \Phi + x^2 \lambda \]  

(4.1-7)

where the Hamiltonian is not a function of \( \dot{x} \), \( x(t) \) and \( \lambda(t) \) are called the canonical variables. In terms of the Hamiltonian, the first variation of Eq. (4.1-1), which is Eq. (4.1-6), becomes

\[ \delta J = -\delta x^2(t_r) \lambda(t_r) + H[x(t_r), \lambda(t_r), t_r] \delta t_r \]  

(4.1-8)

To establish a necessary condition for a minimum, it is necessary that the integrand in Eqs. (4.1-6) and (4.1-8) vanish and also that the transversality condition, as obtained from Eq. (4.1-8)

\[ -\delta x^2(t_r) \lambda(t_r) + H[x(t_r), \lambda(t_r), t_r] \delta t_r = 0 \]  

(4.1-9)

be satisfied.

Thus far in our development we have considered functions with “smooth” arcs. Let us now consider the problem of minimizing the cost function

\[ J = \int_{t_0}^{t_f} x^2(2 - x)^2 \, dt \]

subject to

\[ x(0) = 0, \quad x(1) = 1 \]

Physically, it is clear that the absolute minimum for \( J \) is 0 and that this is obtained for

\[ x(t) = 0 \quad t \in [0, \frac{1}{2}] \]

\[ x(t) = 2t - 1 \quad t \in [\frac{1}{2}, 1] \]

which is certainly a solution to the Euler-Lagrange equation for this problem

\[ x^4 \ddot{x} + x^3 \dddot{x} - 4x = 0 \]

There is one disturbing feature about this solution, however, in that the optimum \( x(t) \) has a “corner” or discontinuous first derivative which gives rise to formal difficulty since \( x \) is contained in the Euler-Lagrange equations. Certainly, though, this particular function \( x(t) \) is smooth in a piecewise sense, or piecewise smooth. We will define a function as being smooth in an interval of time if it is continuous and has a continuous time derivative in the interval. A function is piecewise smooth if it is smooth except for, at most, a finite number of points. We may examine further the special requirements imposed by this “corner” by considering the Weierstrass-Erdmann conditions [1].

The Weierstrass-Erdmann corner conditions furnish us with the requirements for a solution at corners or jumps in the extremal curve. In all of our work thus far (except Section 3.8), we have considered functions defined for
solution of \( J_2(x) \) is of the form \( o(x) \). Since \( x(t) \) satisfies the Euler-Lagrange equations for an extremal and since it is continuous for \( t \in [a, b] \), the function \( x(t) \) must satisfy the Euler-Lagrange equations for a minimum

\[
\frac{d}{dt} \frac{\partial \Phi}{\partial \dot{x}} - \frac{\partial \Phi}{\partial x} = 0
\]

We may rewrite the cost function as a sum of two cost functions:

\[
J(x) = \int_a^b \Phi[x(t), \dot{x}(t), t] \, dt + \int_c^b \Phi[x(t), \dot{x}(t), t] \, dt = J_1(x) + J_2(x)
\]

We may now take the first variation \( \delta J_1(x) \) and \( \delta J_2(x) \) separately. We assume, for the moment only, that \( a \) and \( b \) are fixed, and we require that the \( x(t) \) calculated from \( J_1(x) \) and \( J_2(x) \) is the same at \( t = c \) which is unknown. Since \( c \) is arbitrary, the first variation of \( J(x) \) is

\[
\delta J(x) = \delta x^x(a) \frac{\partial \Phi}{\partial x(a)} + \left\{ \Phi[x(c), x(c), c] - \dot{x}(c) \frac{\partial \Phi}{\partial x(c)} \right\} \delta c
\]

\[
+ \delta x^x(c) \frac{\partial \Phi}{\partial x(c)} + \int_c^b \delta x^x(t) \frac{\partial \Phi}{\partial x(t)} \, dt
\]

Since \( x(t) \) satisfies the Euler-Lagrange equations for an extremal and since \( \delta x(a) = 0 \), we have

\[
\delta J_1(x) = \delta x^x(t) \frac{\partial \Phi}{\partial \dot{x}(t)}
\]

\[
+ \left\{ \Phi[x(t), \dot{x}(t), t] - \dot{x}(t) \frac{\partial \Phi}{\partial \dot{x}(t)} \right\} \delta t
\]

(for \( t = c - 0 \))

In a similar fashion, we can show that the first variation for the extremal solution of \( J(x) \) is

\[
\delta J_2(x) = -\delta x^x(t) \frac{\partial \Phi}{\partial \dot{x}(t)}
\]

\[
+ \left\{ \Phi[x(t), \dot{x}(t), t] - \dot{x}(t) \frac{\partial \Phi}{\partial \dot{x}(t)} \right\} \delta t
\]

(for \( t = c + 0 \))

\[
\delta J(x) = \delta J_1(x) + \delta J_2(x) = 0
\]

\[
\delta \Phi \bigg|_{t'=0} = \frac{\partial \Phi}{\partial x} \bigg|_{t'=0}
\]

\[
\Phi - \dot{x} \frac{\partial \Phi}{\partial x} \bigg|_{t'=0} = \Phi - \dot{x} \frac{\partial \Phi}{\partial x} \bigg|_{t'=0}
\]

since \( \delta x \) and \( \delta t \) are arbitrary. These requirements, Eqs. (4.1-17) and (4.1-18), are called the Weierstrass-Erdmann corner conditions and must hold at any point \( c \) where the extremal has a corner. If we use the Hamiltonian canonical variables

\[
H = \Phi - \dot{x} \frac{\partial \Phi}{\partial x} = \Phi + \lambda \dot{x}
\]

\[
\lambda = -\frac{\partial \Phi}{\partial x}
\]

we immediately see that the Weierstrass-Erdmann conditions simply require \( H \) and \( \lambda \) to be continuous on the optimum trajectory at all points where there are corners.

It is possible to generalize the Weierstrass-Erdmann corner condition in terms of the Weierstrass \( E \) function, defined as

\[
E = \left\{ \Phi(x(t), x(t), t) - \Phi(x, x(t), t) - \dot{x}(t) \frac{\partial \Phi}{\partial x} \right\} \geq 0
\]

where \( \partial \Phi/\partial x \) is evaluated at the optimum solution vector \( x(t) \) and \( \dot{x} \) is an admissible vector, one which satisfies all constraints. This provides us with necessary conditions for an extremum under constrained conditions [1, 6].\(^1\)

In the next section, we will examine, among other things, minimum time problems for problems where the extremal arcs or trajectories are smooth but where the terminal time is not fixed. Thus we will need to use the expanded variational notation presented in the first part of this section. Then we will consider the important case in optimal control where the admissible control and state variables are restricted. We will then use the Weierstrass \( E \) function to develop a maximum principle. In this work we will find it necessary to interpret the vector \( x \) in this section as the generalized state vector, which includes the control vector.

### 4.2 The Bolza problem and its solution

We will introduce the Hamiltonian approach to the solution of variational problems by considering the Bolza problem of the variational calculus and

\[^1\text{Certain other conditions are also required, such as absence of conjugate points. References [1], [6], and [11] provide much elaboration on this point.}\]
several exterior is. We shall see that the results obtained are similar in many ways to the results of the Pontryagin maximum principle which we will present in the next section. Our approach to this section will be, as before, to employ classical variational techniques.

4.2-1. Continuous optimal control problems—fixed beginning
and terminal times—no inequality constraints

We are given a nonlinear differential system operating over the fixed interval \( t \in [t_0, t_f] \) of the form

\[
\dot{x} = f(x, u, t) \tag{4.2-1}
\]

where \( x(t) \), the \( n \) vector state variable, is determined by \( u(t) \), the \( m \) vector control variable, and the initial condition vector

\[
x(t_0) = x_0
\]

Actually, the statement that all components of the \( n \)-dimensional state vector are fixed at the initial time, \( t_0 \), is a bit restrictive, although it is generally true for optimal control problems. However, in the state and parameter estimation problem, not all of the components of the state vector are specified initially. Thus a more general statement of the specified initial conditions is

\[
M(t_0)x(t_0) = m_0 \tag{4.2-3}
\]

where \( m_0 \) is an \( r \) vector. In a similar fashion, some of the terminal states may be specified. In this case, we may find†

\[
N(t_f)x(t_f) = n_f \tag{4.2-4}
\]

where \( n_f \) is a \( q \) vector, \( q \leq n \).

We will return to a discussion of this point momentarily. But now we desire to determine the control \( u(t) \) such as to minimize

\[
\frac{\partial J}{\partial t} + \int_{t_0}^{t_f} \phi(x(t), u(t), t) \, dt \tag{4.2-5}
\]

We use the method of Lagrange multipliers discussed in the last chapter to adjoin the system differential equality constraint to the cost function, which gives us

\[
J = \theta[x(t), t] \mid_{t_0}^{t_f} + \int_{t_0}^{t_f} \phi(x(t), u(t), t) \, dt \tag{4.2-6}
\]

We define a scalar function, the Hamiltonian, as

\[
H[x(t), u(t), \lambda(t), t] = \phi(x(t), u(t), t) + \lambda(t)f[x(t), u(t), t] \tag{4.2-7}
\]

Thus the cost function becomes

\[
J = \theta[x(t), t] \mid_{t_0}^{t_f} + \int_{t_0}^{t_f} [H[x(t), u(t), \lambda(t), t] - \lambda(t)f[x(t), u(t), t]] \, dt \tag{4.2-8}
\]

If we integrate the last term in the integrand of Eq. (4.2-8) by parts, we obtain

\[
J = [\theta[x(t), t] - \lambda(t)f[x(t), u(t), t]] \mid_{t_0}^{t_f} + \int_{t_0}^{t_f} \lambda(t)f[x(t), u(t), t] + \lambda(t)v \, dt \tag{4.2-9}
\]

We now take the first variation of \( J \) for variations in the control vector and, consequently, in the state vector about the optimal control and optimal state vector. This gives us

\[
\delta J = \left\{ \frac{\partial \theta}{\partial x} - \lambda \right\} \mid_{t_0}^{t_f} + \int_{t_0}^{t_f} \left( \frac{\partial \phi}{\partial x} + \lambda f \right) + \delta u \left( \frac{\partial f}{\partial u} \right) \, dt \tag{4.2-10}
\]

A necessary condition for a minimum is that the first variation in \( J \) vanish for arbitrary variations \( \delta x \) and \( \delta u \). Thus we have as the necessary condition for a minimum the very important relations

\[
\left\{ \frac{\partial \theta}{\partial x} - \lambda \right\} = 0, \quad \text{for} \quad t = t_0, t_f \tag{4.2-11}
\]

\[
\frac{\partial \theta}{\partial u} = 0, \quad \lambda = -\frac{\partial H}{\partial x}, \quad \dot{x} = f(x, u, t) = \frac{\partial H}{\partial \lambda} \tag{4.2-12}
\]

Since Eqs. (4.2-3) and (4.2-4), or alternate and perhaps more general expressions for the terminal manifold, may interrelate the components of the vector variation \( \delta x \) at the terminal time, and since an initial manifold may interrelate the components of the vector variation \( \delta x \), Eq. (4.2-11) is the general statement for the transversality condition for the problem treated here. For a large class of optimal control problems, the initial state of the system is specified but the terminal state is unspecified. In that case, Eq. (4.2-11) yields the transversality conditions as

\[
\dot{x}(t_0) = 0, \quad x(t_0) = x_{o_0} \tag{4.2-13}
\]

since \( \delta x(t_0) = 0 \), \( x(t_0) \) is fixed, and \( \delta x(t_f) \) is completely arbitrary. In another broad class of problems \( x(t_f) \) and \( x(t) \) are fixed. In this case \( \delta x(t) \) and \( \delta x(t_f) \) must be zero, and \( x(t_f) \) and \( x(t) \) are the boundary conditions for the two-point boundary value problem. For many estimation problems, neither \( x(t) \) nor \( x(t_f) \) are fixed (specified). In that case, Eq. (4.2-11) yields \( \lambda(t_f) = 0 \) as the boundary conditions for the problem since \( \delta x(t_f) \) and \( \delta x(t) \) are arbitrary. In still another case, we might have \( x(t_0) = x_{o_0} \) and \( \theta = 0 \),
and \( \| \mathbf{x}(t_f) \| = 1 \). In this event, it is easy for us to show that the final transversality conditions are obtained if we solve the two scalar equations, each in \( n \) variables.

\[
\delta \mathbf{x}(t_f) \mathbf{x}(t_f) = 0, \quad \delta \mathbf{x}(t_f) \lambda(t_f) = 0 \tag{4.2-15}
\]

We now give a more general and precise interpretation to the transversality conditions. For the general case where the initial manifold is

\[
\mathbf{M}[x(t_0), t_0] = 0 \quad \text{and} \quad \mathbf{N}[x(t_f), t_f] = 0 \tag{4.2-16}
\]

and the terminal manifold is

\[
\mathbf{N}[x(t_f), t_f] = 0 \tag{4.2-17}
\]

we adjoin these conditions to the \( \theta \) function by means of Lagrange multipliers, \( \xi \) and \( \nu \) and obtain for the cost function

\[
J = \theta[x(t), t] \bigg|_{t=t_f} - \xi^\top \mathbf{M}[x(t_0), t_0] + \nu^\top \mathbf{N}[x(t_f), t_f] + \int_{t_0}^{t_f} \{ \mathbf{H}[x(t), u(t)] + \lambda(t) \mathbf{F}(t) \mathbf{x} \} dt \tag{4.2-18}
\]

We now apply the usual variational techniques to obtain for the transversality conditions at the initial time:

\[
\lambda(t_0) = \frac{\partial \theta}{\partial x} + \left( \frac{\partial \mathbf{M}^\top}{\partial x} \right) \xi, \quad \mathbf{M}[x(t), t] = 0 \quad t = t_0 \tag{4.2-19}
\]

The \( n \) initial conditions are obtained from this, with \( p \) parameters to be found in Eq. (4.2-19) such that we satisfy the \( r \) conditions of Eq. (4.2-16). In a similar fashion, the terminal condition is

\[
\lambda(t_f) = \frac{\partial \theta}{\partial x} + \left( \frac{\partial \mathbf{N}^\top}{\partial x} \right) \nu, \quad \mathbf{N}[x(t), t] = 0, \quad t = t_f \tag{4.2-20}
\]

\( n \) terminal conditions are obtained from this with \( q \) parameters \( \nu \) found in Eq. (4.2-20) such that the \( q \) conditions of Eq. (4.2-17) are satisfied.

The \( n \) vector differential equation obtained from Eq. (4.2-12) will be called the adjoint equation. Equation (4.2-13) provides the coupling relation between the original plant dynamics, Eq. (4.2-1), and the adjoint equation, the \( \lambda \) equation of Eq. (4.2-12). This coupling equation was obtained from

\[
\delta J = \left. \right|_{t=t_0}^{t=t_f} \int_{t_0}^{t_f} \left[ \delta \mathbf{x}^\top \frac{\partial \mathbf{H}}{\partial u} + \delta \mathbf{F}(t) \mathbf{x} \right] dt \tag{4.2-21}
\]

and it is important to note that \( \delta u \) must be completely arbitrary in order for us to draw the conclusion that \( \partial H/\partial u = 0 \) to obtain the optimal control. For the problem posed here where the admissible control set is infinite, \( \delta u \) can be completely arbitrary. Where the admissible control is bounded, \( \delta u \) cannot be completely arbitrary, and \( \partial H/\partial u = 0 \) may not be the correct requirement. We will have more to say about this later. The solution we have obtained for this problem is a special case of the Pontryagin maximum principle.

It is also interesting to note that, since \( H = \phi + \lambda \mathbf{F} \), we may compute the total derivative with respect to time as

\[
\frac{dH}{dt} = \frac{d\phi}{dt} + \dot{x}^\top \left[ \frac{\partial \phi}{\partial x} + \left( \frac{\partial \mathbf{F}}{\partial x} \right) \lambda \right] + \dot{u}^\top \left[ \frac{\partial \phi}{\partial u} + \left( \frac{\partial \mathbf{F}}{\partial u} \right) \lambda \right] + \lambda \mathbf{F} \frac{\partial \mathbf{F}}{\partial u} \tag{4.2-22}
\]

but from Eqs. (4.2-12) and (4.2-7), we have

\[
\lambda = -\frac{\partial H}{\partial x} - \frac{\partial \phi}{\partial x} \left( \frac{\partial \mathbf{F}}{\partial x} \right) \lambda \tag{4.2-23}
\]

and from Eq. (4.2-7),

\[
\frac{\partial H}{\partial u} = \frac{\partial \phi}{\partial u} + \left( \frac{\partial \mathbf{F}}{\partial u} \right) \lambda \tag{4.2-24}
\]

We see that, if \( \phi \) and \( \mathbf{F} \) are not explicit functions of time, the Hamiltonian is constant along an optimal trajectory where \( \partial H/\partial u = 0 \). It can be shown that this is always true along an optimal trajectory, even if we cannot require \( \partial H/\partial u = 0 \). We will make use of this fact in a later development.

In order that \( J \) be a minimum, the second variation of \( J \) must be nonnegative along all trajectories such that Eq. (4.2-1) is satisfied. Therefore we need to compute the second variation of \( J \) in Eq. (4.2-9) and impose the requirement that the variation of Eq. (4.2-1) is zero, or that

\[
\delta \mathbf{x} - \left( \frac{\partial \mathbf{F}}{\partial u} \right) \delta \mathbf{u} = 0 \tag{4.2-25}
\]

Applying this condition and taking the quadratic part of the Taylor series expansion of \( J(x + \delta x, u + \delta u) - J(x, u) \), Eq. (4.1-4), we have for the second variation

\[
\delta^2 J = \left. \right|_{t=t_0}^{t=t_f} \int_{t_0}^{t_f} \left[ \delta \mathbf{x}^\top \frac{\partial^2 \phi}{\partial x^2} \delta \mathbf{x} + \left( \frac{\partial \mathbf{F}}{\partial x} \right)^2 \lambda \right] dt + \left. \right|_{t=t_0}^{t=t_f} \int_{t_0}^{t_f} \left[ \delta \mathbf{x}^\top \frac{\partial \mathbf{F}}{\partial u} \delta \mathbf{u} + \left( \frac{\partial \mathbf{F}}{\partial u} \right)^2 \lambda \right] dt \tag{4.2-26}
\]

and this must be nonnegative for a minimum. This will be the case if the \( n + m \) square matrix under the integral sign and \( \partial^2 \phi/\partial x^2 \) are nonnegative definite.
Example 4.2-1

We are given the differential system consisting of three cascaded integrators

\[
\begin{align*}
\dot{x}_1 &= x_2, & x_1(0) &= 0 \\
\dot{x}_2 &= x_3, & x_2(0) &= 0 \\
\dot{x}_3 &= u, & x_3(0) &= 0
\end{align*}
\]

We wish to drive the system so that we reach the terminal manifold

\[x(1) + x_3(1) = 1\]

such that the cost function

\[J = \int_i^f u^2 \, dt\]

is minimized. The solution to the problem proceeds as follows. We compute the Hamiltonian from Eq. (4.2-7) as

\[H = \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u\]

and determine the coupling relation, Eq. (4.2-13),

\[\frac{\partial H}{\partial u} = 0 = u + \lambda_3\]

and the adjoint Eq. (4.2-12)

\[\lambda_1 = -\frac{\partial H}{\partial x_1} = 0\]
\[\lambda_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1\]
\[\lambda_3 = -\frac{\partial H}{\partial x_3} = -\lambda_2\]

From Eqs. (4.2-17) and (4.2-20) we see that the transversality condition at the terminal time is

\[x(1) + x_3(1) = 1\]

\[\lambda(1) = \frac{\partial \theta}{\partial x} + \left( \frac{\partial N^T}{\partial x} \right) \nu, \quad t = t_f\]

where

\[N[x(t_f), t_f] = x(t_f) + x_3(t_f) - 1 = 0, \quad t_f = 1\]

Thus

\[\lambda(1) = \begin{bmatrix} \lambda_1(1) \\ \lambda_2(1) \\ \lambda_3(1) \end{bmatrix} = \begin{bmatrix} 2x_1(1) \nu \\ 2x_2(1) \nu \\ 0 \end{bmatrix}\]

Thus the problem of finding the optimal control and associated trajectories for this example is completely resolved when we solve the two-point boundary value problem represented by

\[\dot{x}_1 = x_2, \quad x_1(0) = 0\]
\[\dot{x}_2 = x_3, \quad x_2(0) = 0\]
\[\dot{x}_3 = u, \quad x_3(0) = 0\]

Although the six first-order differential equations represented above are perfectly linear and time invariant, the solution to this problem is complicated by the nonlinear nature of the terminal conditions. We shall discover various iterative schemes for overcoming this difficulty in later chapters.

4.2-2. Continuous optimal control problems—fixed beginning and unspecified terminal times—no inequality constraints

The material of the previous subsection may be easily extended to the case where the terminal manifold equation is a function of the terminal time and the terminal time is unspecified. For convenience, we will assume that the initial time and the initial state vector are specified. Solution may then easily be obtained for the case where the initial time and initial state vector are unspecified. Therefore the problem becomes one of minimizing the cost function

\[J = \theta[x(t_f), t_f] + \int_i^{t_f} \phi(x(t), u(t), t) \, dt\] (4.2-27)

for the system described by

\[\dot{x} = f[x(t), u(t), t], \quad x(t_i) = x_0\]

where \(t_i\) is fixed and where, at the unspecified terminal time \(t = t_f\), the \(q\) vector terminal manifold equation

\[N[x(t_f), t_f] = 0\] (4.2-29)

is satisfied. It may be noted here that the terminal manifold line, \(x(t_f) = c(t_f)\), of the previous chapter becomes here \(N[x(t_f), t_f] = 0\) which is more general. We adjoin the equality constraints to the cost function via Lagrange multipliers to obtain

\[J = \theta[x(t_f), t_f] + \nu^T N[x(t_f), t_f] + \int_i^{t_f} \phi[x(t), u(t), t] + \lambda^T[f[x(t), u(t), t] - \dot{x}] \, dt\] (4.2-30)

As before, we define the Hamiltonian

\[H[x(t), u(t), \lambda(t), t] = \phi[x(t), u(t), t] + \lambda^T f[x(t), u(t), t]\]

and integrate a portion of the cost function, Eq. (4.2-30), to obtain

\[J = \theta[x(t_f), t_f] + \nu^T N[x(t_f), t_f] - \lambda^T(t_f) x(t_f) + \lambda^T(t_i) x(t_i) + \int_i^{t_f} [H[x(t), u(t), \lambda(t), t] + \lambda^T(\dot{x})] \, dt\] (4.2-31)
We again obtain the first variation by letting

\[ x(t) = \hat{x}(t) + h(t), \quad u(t) = \hat{u}(t) + h(t), \quad t_f = t_f + \delta t_f \quad (4.2-32) \]

and then we form the difference \( J[x, u, t_f] - J[\hat{x}, \hat{u}, t_f] \) and retain only the linear terms. Thus we have, after dropping the \( \Theta \) notation for convenience,

\[
\delta J = \delta t_f \left\lbrace \frac{\partial H}{\partial x} \left[ \dot{x}(t_f) \right] + \frac{\partial H}{\partial u} \left[ u(t_f) \right] \right\}
+ \delta x^T(0) \left[ \frac{\partial H}{\partial x} \right] \dot{x}(0) + \delta u^T(0) \left[ \frac{\partial H}{\partial u} \right] u(0)
+ \int_0^{t_f} \left[ \frac{\partial H}{\partial x} \left[ \dot{x}(t) \right] + \frac{\partial H}{\partial u} \left[ u(t) \right] \right] dt
\]

where

\[
\Theta[x(t_f), y, t_f] = \left[ \frac{\partial H}{\partial x} \right] \dot{x}(t_f) + \left[ \frac{\partial H}{\partial u} \right] u(t_f)
\]

(4.2-33)

We must set this first variation equal to zero to obtain the necessary conditions for a minimum. Therefore, the equations which determine the optimal control and state vector are

\[
H = \phi[x(t), u(t), t] + \lambda^T(t) f[x(t), u(t), t]
\]

(4.2-35)

\[
\frac{\partial H}{\partial x} = \dot{x} = f[x(t), u(t), t]
\]

(4.2-36)

\[
\frac{\partial H}{\partial u} = \lambda = \frac{\partial \phi}{\partial x}[x(t), u(t), t]
\]

(4.2-37)

\[
\frac{\partial H}{\partial \dot{x}} = 0 = \frac{\partial \phi}{\partial x}[x(t), u(t), t] + \frac{\partial \phi}{\partial u}[x(t), u(t), t] \lambda[t]
\]

(4.2-38)

These represent the 2n differential equations for the two-point boundary value problems. The conditions at the initial time are

\[ x(t_0) = x_o \]

(4.2-39)

whereas those at the final time are

\[ \lambda(t_f) = \phi[x(t_f), t_f] + \left[ \frac{\partial \phi}{\partial x} \right] x(t_f) = 0 \quad (4.2-40) \]

\[ N[x(t_f), t_f] = 0 \quad (4.2-41) \]

and

\[ H[x(t_f), u(t_f), \lambda(t_f), t_f] + \frac{\partial H}{\partial t_f} = 0 \quad (4.2-42) \]

Equation (4.2-40) provides \( n \) conditions with \( q \) Lagrange multipliers to be determined. Equation (4.2-41) provides \( q \) equations to eliminate the Lagrange multipliers, and Eq. (4.2-42) provides the one additional equation which we must have to determine the unspecified terminal time.

**Example 4.2-2**

For the first-order single integration system

\[ \dot{x} = u, \quad x(0) = 1 \]

de we desire to find the control \( u(t) \) which makes \( x(t_f) = 0 \), where \( t_f \) is unspecified, such as to make, for specified values of \( \alpha \) and \( \beta \),

\[ J = t_f^2 + \beta \int_0^{t_f} u^2 dt \]

(4.2-34)

a minimum. For this problem

\[ N[x(t_f), t_f] = \lambda(t_f) = 0, \quad \phi = \frac{1}{2} \beta u^2 \]

(4.2-35)

\[ t_f = \gamma, \quad H = \frac{1}{2} \beta u^2 + \lambda(t) \lambda(t) \]

The canonic equations are

\[ \dot{x} = u = \frac{-\lambda}{\beta}, \quad \lambda = 0 \]

with the boundary conditions \( x(0) = 0, x(t_f) = 0 \), where we determine the final time by solving Eq. (4.2-42) which becomes, for this example,

\[ \frac{-\lambda^2(t_f)}{2 \beta} + \alpha t_f^{-1} = 0 \]

(4.2-36)

The solutions to the canonic equations are

\[ x(t) = -\frac{\lambda(t_f)}{\beta} t + 1, \quad \lambda(t) = \lambda(t_f) \]

(4.2-37)

But since \( x(t_f) = 0; t_f = \beta x^{-1}(t_f) \), and in the particular case where \( \beta = \alpha = 1 \),

| \[ x(t_f) = \beta x^{-1}(t_f) \] | (4.2-38) |

we can easily show from the foregoing that \( \lambda(t_f) + 2(2)^{1/2} \), which determines the solution to this example. The optimum control is \( u(t) = -\lambda(t) = -2^{1/2} \). The corresponding trajectory is \( x(t) = 1 - 2^{1/2} t \), with \( t_f = 2^{-1/2} \).

**Example 4.2-3**

A problem which will be of considerable interest to us later will be the "minimum time" problem. In that case

\[ \theta[x(t_f), t_f] = t_f, \quad \phi = 0 \]

and we specify the optimal control and corresponding trajectory by solving Eqs. (4.2-35) through (4.2-38), which become

\[ H[x(t), u(t), \lambda(t), t] = \lambda^T(t) f[x(t), u(t), t] \]

\[ \frac{\partial H}{\partial x} = \dot{x} = f[x(t), u(t), t] \]

\[ \frac{\partial H}{\partial u} = \lambda = \frac{\partial \phi}{\partial x}[x(t), u(t), t] \lambda(t) \]

\[ \frac{\partial H}{\partial \dot{x}} = 0 = \frac{\partial \phi}{\partial x}[x(t), u(t), t] \lambda(t) \]

(4.2-39)
In many cases, the system is brought to rest at the unspecified time, and the terminal manifold is the origin, so that

\[ N[x(t_f), t_f] = 0 \]

Then the foregoing expressions reduce to

\[ x(t_f) = x_n, \quad u(t_f) = 0 \]

\[ H[x(t_f), u(t_f), \lambda(t_f), t_f] = -1 \]

If the Hamiltonian is not an explicit function of time, Eq (4.2-24), which applies here as well, yields \( dH/dt = 0 \); therefore, for this minimum time problem

\[ H[x(t), u(t), \lambda(t), t] = H[x(t), u(t)] = -1 \]

It should be emphasized that we are not solving the usual minimum time problem since we have imposed no inequality constraints on the control (or state) variables. An alternate version of this problem would be to consider the variable end-point and variable end-time case.

4.3 The Bolza problem with control and state variable inequality constraints—the Pontryagin maximum principle

In the prior work in this chapter we treated the Bolza problem with no inequality constraints present on either the control or the state variable. We found for example that a minimum of

\[ J = \theta[x(t_f), t_f] + \int_{t_0}^{t_f} \phi[x(t), u(t), t] dt \]

for a system described by

\[ \dot{x} = f[x(t), u(t), t], \quad x(t_0) = x_0 \]

with \( t_0 \) and \( t_f \) fixed may be obtained if we define a Hamiltonian as

\[ H[x(t), u(t), \lambda(t), t] = \phi[x(t), u(t), t] + \lambda(t) f[x(t), u(t), t] \]

and set

\[ \frac{\partial H}{\partial \lambda} = \dot{\lambda}, \quad x(t_0) = x_0 \]

with the boundary conditions specified by Eqs. (4.2-39) through (4.2-42)

\[
\begin{align*}
\dot{x}(t_f) &= x_0, \\
\lambda(t_f) &= \frac{\partial N}{\partial x}(t_f) v \\
N[x(t_f), t_f] &= 0 \\
H[x(t_f), u(t_f), t_f] &= -1 - \left( \frac{\partial N}{\partial t_f} \right) v
\end{align*}
\]

If the admissible control vector is unrestricted, then the first variation of \( H \), with \( u(t) \), is also unrestricted, and in that part of Eq. (4.2-10) which reads

\[ \int_{t_0}^{t_f} [\delta u(t)]^T \left[ \frac{\partial H}{\partial u} \right] dt + \cdots = 0 \]

we are free to set \( \partial H/\partial u \) equal to zero. Sections 4.1 and 4.2 describe a special case of the maximum principle where this is possible. In many problems, inequality constraints on the admissible control vector (the maximum thrust from a reaction jet is limited, for example) are present, and we must therefore take this into account if we are to determine a realistic control strategy. If \( u(t) \) is constrained, \( \delta u(t) \) may not be allowed to be completely arbitrary, and therefore we may not in general set \( \partial H/\partial u = 0 \). Also, certain regions of the state space may be prohibited, and we must determine an optimum control such that the state \( x(t) \) does not enter the forbidden regions. We examined a portion of this problem in Chapter 3 and found that we could handle inequality constraints by converting them to equivalent equality constraints. In this section, we desire to find the state and control vector such that the cost function

\[ J = \theta[x(t_f), t_f] + \int_{t_0}^{t_f} \phi[x(t), u(t), t] dt \]  

(4.3-1)

is minimized subject to

(a) the \( n \) differential system equality constraints

\[ \dot{x} = f[x(t), u(t), t] \]  

(4.3-2)

(b) the \( q \) end point equality constraints \( (q \leq n) \) at the terminal time (which may be unspecified)

\[ N[x(t_f), t_f] = 0 \]  

(4.3-3)

and the initial condition equality constraint

\[ x(t_0) = x_0 \]  

(4.3-4)

where we assume that \( t_f \) is fixed and \( x(t_f) \) is known. Actually, \( t_f \) does not have to be fixed and the initial condition constraint can be \( M[x(t_0), t_0] = 0 \), as was the case in Section 4.2. The required modifications to treat this case are small since the results are so similar to the variable end-point and variable end-time case.

(c) The \( r \) admissible control inequality constraints \( (r \leq m) \)

\[ g[x(t), u(t), t] \geq 0 \]  

(4.3-5)
where we will find it necessary to impose the requirement that the ma
\[ \frac{\partial g}{\partial u} \] be of maximum rank whenever \( g = 0. \)

d) The \& inequality constraints (with no component in the
constraint) expressing the forbidden region of state space
\[ h(x(t), t) \geq 0 \quad (4.3-6) \]

which does not satisfy the maximum rank test in (c).

As is apparent, we have formulated a rather formidable problem in the
variational calculus. We will solve the problem in such a fashion that we
obtain the Pontryagin maximum principle [2, 3, 4, 5]. However, due to a
slight change in the original problem statement, a more appropriate name for
the result of our development would be the Pontryagin minimum principle.
Our development will be patterned after that of Berkovitz who has unified
many of the approaches to the optimal control problem [6, 7]. We will first
consider the case where the inequalities of part (d) on the admissible regions
of state space are not present and will then modify our maximum principle
and associated transversality conditions to include this important case.

4.3-1. The maximum principle with control variable inequality
constraints

We now wish to derive the first necessary condition for a minimum of
the problem just posed, except that we will assume that there are no bounded
state variables. Thus we are considering the first three of the four constraints
just mentioned. Constraint (c) is very similar to the inequality constraint of
Section 3.8, and we now find it desirable to expand upon that method of
treating an inequality constraint.

We are given the inequality constraint
\[ g[x(t), u(t), t] \geq 0 \quad (4.3-7) \]

We may convert this inequality constraint to an equality constraint by
writing for each component of \( g \) either
\[ (\lambda_i)^t = g_i[x(t), u(t), t], \quad \lambda_i(t_0) = 0, \quad i = 1, 2, \ldots, r \quad (4.3-8) \]
or
\[ (\gamma_i)^t = g_i[x(t), u(t), t], \quad \gamma_i(t_0) = 0, \quad i = 1, 2, \ldots, r \quad (4.3-9) \]

It is apparent that either of these two equations force \( \lambda \) to be greater
than or equal to zero since \( (\lambda_i)^t \) and \( (\gamma_i)^t \) must certainly be greater than or
equal to zero. This technique was apparently first proposed by Valentine [8]
and extended by Berkovitz [6]. It is quite similar to the penalty function
technique of Kelly [9] as we shall see in our chapter concerning the gradient
and second variation methods for the computation of optimal controls.
The choice between Eqs. (4.3-8) and (4.3-9) will depend largely upon the
particular computer (for an analog computer, Eq. (4.3-8) is generally easier
to implement than Eq. (4.3-9) and the particular computational algorithms
used for the quasilinearization method, Eq. (4.3-9) is considerably simpler
to use than Eq. (4.3-8) and also results in less computer solution time).

Example 4.3-1

It is quite easy to see that the constraint used here includes, as a special case,
that considered in Section 3.8. For example, if we require for a scalar control \( u \),
\[ u_{\min} \leq u \leq u_{\max} \] then we may write
\[ g_1[x(t), u(t), t] = u_{\max} - u \geq 0, \quad g_2[x(t), u(t), t] = u - u_{\min} \geq 0 \]

and we convert these inequality constraints to equality constraints by writing
\[ (\gamma_1)^2 = u_{\max} - u, \quad (\gamma_2)^2 = u - u_{\min} \]

for which
\[ (\gamma_1, \gamma_2)^2 = (u_{\max} - u)(u - u_{\min}) \]

which is precisely the constraint used in Section 3.8.

For the problem at hand, we adjoin, via the Lagrange multiplier, con-
straints (4.3-2), (4.3-3), (4.3-4), and (4.3-5) to Eq. (4.3-1) to obtain
\[ J = \rho[x(t_f), t_f] + \nu^rN[x(t_f), t_f] \]
\[ + \int_{t_0}^{t_f} \left[ H[x(t), \dot{w}(t), \lambda(t), t] - \lambda(t) \dot{x} \right] dt \]
\[ - \int_{t_0}^{t_f} \left[ g_1[x(t), w(t), t] \right. dt \] (4.3-10)

where
\[ (z^*)^r = \{ z_1^*, z_2^*, \ldots, z^*_r \} \quad (4.3-11) \]
\[ H[x(t), \dot{w}(t), \lambda(t), t] = \rho[x(t), \dot{w}(t), t] + \lambda(t) \dot{F}[x(t), \dot{w}(t), t] \]
\[ \dot{w} = u(t), \quad w(t_0) = 0 \quad (4.3-12) \]

We may now apply the Euler-Lagrange equations to the above cost
function or take a first variation of it in order to obtain the necessary condi-
tions for a minimum. It is thus convenient to define a scalar function \( \Phi \),
the Lagrangian,
\[ \Phi[x(t), \dot{x}(t), \dot{w}(t), \lambda(t), \lambda(t), \Gamma(t), \dot{z}(t), t] = H[x(t), \dot{w}(t), \lambda(t), t] \]
\[ - \lambda(t) \dot{x} - \Gamma(t) [g_1[x(t), w(t), t] - 2^*] \] (4.3-14)

We will use the Euler-Lagrange Eqs. (3.5-3). Since there are no \( w(t) \) and \( z(t) \)
terms in Eq. (4.3-14), we may write the Euler-Lagrange equations as
\[ \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{x}} = 0 \quad (4.3-15) \]
\[ \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{w}} = 0 \quad (4.3-16) \]
\[ \frac{d}{dt} \frac{\partial \Phi}{\partial \dot{z}} = 0 \quad (4.3-17) \]
Each piecewise continuously differentiable solution of the Euler-Lagrange equations (4.3-15), (4.3-16), and (4.3-17) will be called an extremal curve or an extremal trajectory of the associated variational problem. It can be shown that the function \( \Phi \) need be only piecewise smooth, and thus the Euler-Lagrange equations require that every arc of the extremal trajectory on which the first derivatives of \( \Phi \) have no discontinuities be a solution of the Euler-Lagrange equations. The corner condition will answer our questions concerning what happens at possible points of discontinuity of some of the derivatives of the state or control variables. This corner condition will ensure continuity of the state and control variables by forcing \( \partial \Phi / \partial \xi \) to be zero everywhere since it is zero at the terminal time.

The transversality conditions for this problem are obtained in the usual fashion as explained in Chapter 3 and the previous three sections. For this problem, they are easily shown to be Eqs. (4.3-3), (4.3-4), and

\[
\frac{\partial \Phi}{\partial \xi} + (\frac{\partial N}{\partial \xi}) \nu + \phi - \frac{\partial \Phi}{\partial \xi} = 0, \quad \text{for} \quad t = t_f
\]

Also, we have for the final transversality condition

\[
\frac{\partial \Phi}{\partial \omega} + (\frac{\partial N}{\partial \omega}) \nu \lambda = 0, \quad \text{for} \quad t = t_f
\]

which allows us to write because of Eq. (4.3-17)

\[
\frac{\partial \Phi}{\partial \xi} = 0, \quad \forall t \in [t_0, t_f]
\]

Since when \( \Gamma \neq 0 \), \( z_i = 0 = g_i \), and when \( z_i \neq 0, \Gamma = 0 \)

\[
\Gamma \xi = 0, \quad i = 1, 2, \ldots, r, \quad \forall t \in [t_0, t_f]
\]

Also, with similar reasoning, we have

\[
\frac{\partial \Phi}{\partial \omega} = 0, \quad \forall t \in [t_0, t_f]
\]

We shall now introduce the Hamiltonian formulation and use the Weierstrass condition to obtain the Pontryagin maximum principle. From the definition of \( \Phi \), Eq. (4.3-14), Eq. (4.3-15) yields

\[
\dot{x} = -\frac{\partial H}{\partial x} + \Gamma
\]

Equation (4.3-16) with the definition of \( \Phi \), Eq. (4.3-14), gives us

\[
\frac{\partial H}{\partial \omega} - \frac{\partial N}{\partial \omega} \Gamma = 0
\]

and in a similar fashion, Eq. (4.3-17) results in

\[
\Gamma^i_{1j} = 0, \quad i = 1, 2, \ldots, r
\]

Since Eq. (4.3-14), when solved for \( H \), yields

\[
H[x, \omega, \lambda, t] = \Phi(x(t), \omega(t), \lambda(t), \Gamma(t), \dot{x}(t), t) + \lambda^i(t) \dot{x} + \Gamma^j(t)[g(x(t), \omega(t), t) - \dot{z}]
\]

we can show that

\[
H = \Phi - \dot{x} \frac{\partial \Phi}{\partial \xi} - \omega \frac{\partial \Phi}{\partial \omega} - z \frac{\partial \Phi}{\partial z}
\]

because we know that \( \dot{z} = g, \dot{x} = \frac{\partial \Phi}{\partial \xi}, \) and have just found \( \frac{\partial \Phi}{\partial \omega} = 0 \) and \( \frac{\partial \Phi}{\partial z} = 0 \). This is in a form for direct application of the Weierstrass condition, Eq. (4.1-21), which can be written as

\[
\sqrt{c^2} / \sqrt{c^2 - \Phi(x, w, z, \dot{x}, \omega, \lambda, t) - \Phi(x, w, z, \dot{x}, \omega, \lambda, t) - (\dot{x} - \dot{x}) \frac{\partial \Phi}{\partial \xi}} \leq 0
\]

where lower-case symbols indicate optimum vectors and upper-case symbols indicate admissible vectors, as before. From Eq. (4.3-25), it becomes apparent that this condition is equivalent to

\[
H[x, \dot{x}, \lambda, t] \geq H[x, \dot{x}, \lambda, t]
\]

In other words, the Hamiltonian is smaller when we use the optimal control within the admissible set of controls than it is for any other control which is in this admissible set. This is the basic contribution of the maximum principle—a necessary condition for optimality is the global minimization of the Hamiltonian, \( H \), function.

4.3-2. Summary of the maximum principle

Since our development of the maximum principle has been necessarily long, it is desirable to give a summary of the results. It is also important to note that we can successfully use the maximum principle without following each and every detail of our “proof.”

We wish to minimize

\[
J = \theta(x(t_f), t_f) + \int_{t_0}^{t_f} \Phi(x(t), u(t), t) dt
\]

for the system described by

\[
x = f(x(t), u(t), t)
\]

\[
x(t_0) = x_0, \quad t_0 \text{ fixed}
\]

such that, at the unspecified terminal time \( t_f \),

\[
N[x(t_f), t_f] = 0
\]
and where \( t \) is restricted such that
\[
g[u(t), t] \geq 0 \tag{4.3-32}
\]
In other words, \( u(t) \) is not restricted in control space as a function of the state vector, \( x(t) \), and
\[
u \in \mathbb{U} \tag{4.3-33}
\]

The Hamilton canonic equations, solution of which minimizes the cost function and determines the optimum state and control vectors, \( x(t) \) and \( u(t) \), may be obtained if we define a Hamiltonian
\[
H[x(t), u(t), \lambda(t), t] = \phi[x(t), u(t), t] + \lambda^T(t)[f(x(t), u(t), t)] \tag{4.3-34}
\]
and then set the Hamiltonian with \( u = u \) less than any other value of \( H \) with \( u \in \mathbb{U} \).

subject to the two-point boundary conditions
\[
x(t_0) = x_0 \tag{4.3-38}
\]
\[
N[x(t_f), t_f] = 0 \tag{4.3-39}
\]
\[
\frac{\partial N}{\partial t} + \left( \frac{\partial N}{\partial x} \right) v + H = 0, \quad \text{at} \quad t = t_f \tag{4.3-40}
\]
\[
\frac{\partial N}{\partial x} + \left( \frac{\partial N}{\partial x} \right) v - \lambda = 0, \quad \text{at} \quad t = t_f \tag{4.3-41}
\]

We frequently wish to transfer the system to the origin in minimum time so that we have
\[
x(t_f) = x_0 \tag{4.3-42}
\]
\[
\theta[x(t_f), t_f] = t_f \tag{4.3-43}
\]
\[
\phi = 0 \tag{4.3-44}
\]

In this particular case, the transversality conditions become
\[
x(t_0) = x_0 \tag{4.3-45}
\]
\[
x(t_f) = 0 \tag{4.3-46}
\]
\[
H = -1, \quad \text{at} \quad t = t_f \tag{4.3-47}
\]

Example 4.3-2

Let us consider briefly the time optimal control problem for a linear time-invariant system where the length of the control vector is constrained. We wish to minimize
\[
J = t_f
\]
for the system
\[
\dot{x} = Ax(t) + Bu(t) \\
x(t_0) = x_0
\]
where \( u(t) \in \mathbb{U} \) means \( ||u(t)|| \leq 1 \).

The Hamiltonian, Eq. (4.3-34), becomes
\[
H[x(t), u(t), \lambda(t), t] = \lambda^T(t)[Ax(t) + Bu(t)]
\]
To make \( H \) as small as possible with respect to a choice of \( u(t) \), we must have
\[
\frac{\partial H}{\partial x} = \dot{x} = Ax(t) + Bu(t), \quad \frac{\partial H}{\partial x} = -\dot{\lambda} = A^T\lambda(t)
\]

The canonic equations become
\[
\frac{\partial H}{\partial x} = \dot{x} = Ax(t) + Bu(t), \quad \frac{\partial H}{\partial x} = -\dot{\lambda} = A^T\lambda(t)
\]
with the boundary conditions
\[
x(t_0) = x_0, \quad x(t_f) = 0
\]
where we determine \( t_f \) by solving
\[
H[x(t_f), \lambda(t_f), u(t_f)] = -1
\]
But, from Eq. (4.2-24) we see that \( dH/dt = 0 \) since the Hamiltonian does not depend explicitly on \( t \). Thus the above equation becomes
\[
H[x(t_f), u(t_f), \lambda(t_f)] = -1 = \lambda^T(t_f)[Ax(t) + Bu(t)]
\]
which is the additional relation needed to determine the terminal time.

4.3.3 The maximum principle with state (and control) variable inequality constraints

We now wish to extend the work of Section 4.3-1 to include inequality constraints on some or all of the state variables. We will represent this inequality constraint by the \( s \) vector equation
\[
h[x(t), t] \geq 0 \tag{4.3-48}
\]
where each component of \( h \) is assumed to be continuously differentiable in state space. There are several methods whereby we may convert Eq. (4.3-48) to an equality constraint. We may define a new variable \( x_{n+1} \) by
\[
\dot{x}_{n+1} = f_{n+1} = [h_i(x, t)]^T H[h_i] + [h_i(x, t)]^T H[h_i] + \cdots + [h_i(x, t)]^T H[h_i]
\]
where \( H[h_i(x, t)] \) is a modified Heaviside step defined such that
\[
H[h_i(x, t)] = \begin{cases} 
0 & \text{if } h_i(x, t) \geq 0 \\
K_s & \text{if } h_i(x, t) < 0 
\end{cases}
\]
and where the initial condition is
\[
x_{n+1}(t_0) = 0 \tag{4.3-50}
\]
Thus we see that
\[ x_{n+t}(t_f) \] is a direct measure of penetration of the state variable inequality constraint
\[
\int_{t_1}^{t_f} \dot{x}_{n+1}(t) \, dt = \int_{t_1}^{t_f} \left[ [h(x, t)]^2 H(h_i) + \cdots + [h(x, t)]^2 H(h_i) \right] \, dt
\]

We will require that the final value of \( x_{n+1}(t_f) \) is zero,
\[ x_{n+1}(t_f) = 0 \]
which will impose the restriction that we do not violate the inequality constraint. This approach is a modification by McGill \[10\] of a similar procedure by Kelley \[9\] which converts the inequality constraint to equality constraints of the form
\[
\dot{x}_{n+1} = [h(x, t)]^2 H(h_i), \quad x_{n+1}(t_0) = 0
\]
\[
\dot{x}_{n+2} = [h(x, t)]^2 H(h_i), \quad x_{n+2}(t_0) = 0
\]
\[
\vdots
\]
\[
\dot{x}_{n+s} = [h(x, t)]^2 H(h_i), \quad x_{n+s}(t_0) = 0
\]
which are then added to the cost function to obtain
\[
J_{\text{modified}} = J_{\text{original}} + \sum_{j=1}^{s} x_{n+j}(t_f)
\]

The multipliers \( K_i \) are thus the penalty functions, and \( J_{\text{modified}} \) is minimized such that the constraint region is entered only slightly, if at all. If we require \( x_{n+1}(t_f) = 0 \) for \( j = 1, 2, \ldots, s \), the constraint is of course not exceeded at all.

A slight modification of the penalty-function approach can be obtained if we define \( s \) new state variables
\[
(x_{n+1})^3 = K_t h(x, t), \quad x_{n+1}(t_0) = 0
\]
\[
(x_{n+2})^3 = K_t h(x, t), \quad x_{n+2}(t_0) = 0
\]
\[
\vdots
\]
\[
(x_{n+s})^3 = K_t h(x, t), \quad x_{n+s}(t_0) = 0
\]

Berkovitz \[7\] suggests yet another method for converting the inequality constraint to an equality constraint. For the case of a scalar constraint, a variable
\[
\varphi(x, \eta, t) = \begin{cases} 
\eta^2 - h(x, t) & \text{if } \eta > 0 \\
\eta h(x, t) & \text{if } \eta < 0 
\end{cases}
\]
is introduced and we convert the inequality constraint \( h(x, t) \geq 0 \) to an equality constraint by writing
\[
\frac{\partial \varphi \, d\eta}{\partial \eta \, dt} = \frac{\partial h \, dx}{\partial x \, dt} + \frac{\partial h \, dx}{\partial x \, dt}
\]
which satisfies the constraint if we have the end conditions
\[
\gamma(x(t_0), \eta(t_0), t_f) = 0
\]

The Euler-Lagrange equations can, of course, be used to determine the differential equations for an extremum, and the associated transversality conditions can be used to specify the two-point boundary values. If inequality constraints on the control variables are present, we must of necessity incorporate these into our problem formulation. The Hamiltonian formulation may also be used. These methods provide us with necessary conditions only.

From Eq. (4.3-14) it follows that the Lagrangian for the problem at hand is
\[
\dot{\Phi} = \Phi + \lambda \varphi(t, x) - \lambda \varphi(t, x) + \lambda \varphi(t, x)
\]
\[
\dot{\Phi} = H - \lambda \varphi(t, x) - \lambda \varphi(t, x) + \lambda \varphi(t, x)
\]

where \( \Phi \) is the Lagrangian for no inequality state constraint. We are using the equality constraint method of Eqs. (4.3-49) and (4.3-50). The Euler-Lagrange equations yield
\[
\frac{d}{dt} \frac{\partial \Phi}{\partial x} - \frac{\partial f_{n+1}}{\partial x} = 0
\]
\[
\frac{d}{dt} \frac{\partial \Phi}{\partial u} = 0
\]
\[
\frac{d}{dt} \frac{\partial \Phi}{\partial \lambda} = 0
\]
which are, except for the \( f_{n+1} \) term, exactly the same as Eqs. (4.3-15), (4.3-16), and (4.3-17). Also, we see that
\[
\frac{d}{dt} \varphi(t, x) = 0
\]
with the transversality conditions exactly as before and, in addition,
\[
\varphi(t_0) = \varphi(t_f) = 0
\]

It is desirable to reinterpret these results in terms of the Hamiltonian, just as we have done for the case of control variable constraints only. We can do this easily by combining Eq. (4.3-60) with Eq. (4.3-61) and making use of the Weierstrass condition, Eq. (4.3-26), which yields
\[
\dot{\lambda} = \frac{d\lambda(t)}{dt} = \frac{\partial H}{\partial x} - \frac{\partial f_{n+1}}{\partial x} \lambda_{n+1}
\]
\[
\dot{x} = \frac{dx(t)}{dt}
\]
\[
\dot{\lambda} = \frac{d\lambda(t)}{dt}
\]

(4.3-67)

(4.3-68)

(4.3-69)
Example 4.3-3

As an example of optimization with a state variable constraint, we consider the brachistochrone problem previously treated by McGill [10] and Dreyfus [11]. A particle is falling for a specified time, \( t_f - t_o \), under the influence of a constant gravitational acceleration \( g \). The particle has initial velocity \( x_o(t_0) = x_{o0} \). We wish to find the path that maximizes the final value of the horizontal coordinate \( x(t_f) \). The final value of the vertical coordinate \( x(t_f) \) and the velocity \( x(t_f) \) are unspecified. The path is constrained by a line \( h(x_i, x_j) \geq 0 \) in the \( x_i x_j \) plane, where it is known that the unconstrained solution intersects the line.

The system dynamics are described by

\[
\begin{align*}
\dot{x}_1 &= x_3 \cos u, \quad x_1(t_0) = x_{10} \\
\dot{x}_2 &= x_3 \sin u, \quad x_2(t_0) = x_{20} \\
\dot{x}_3 &= g \sin u, \quad x_3(t_0) = x_{30}
\end{align*}
\]

where \( u \) is the slope of the path. The cost function is

\[
J = -x_1(t_f)
\]

with no specified endpoint equality constraints, and the state vector inequality constraint

\[
h(x_i, x_j) = ax_i + b - x_i \geq 0
\]

which is converted to the equality constraint

\[
x_i = f_i = [h(x_i, x_j)]H(h)
\]

We can easily compute the requisite nonlinear two-point boundary value problem by direct application of the maximum principle in Section 4.3.3. The equations of this TPBVP are

\[
\begin{align*}
x_1 &= x_3 \lambda_1 (x_3 + \lambda_3 g)^{1/2}, \quad x_1(t_o) = x_{10} \\
x_2 &= x_3 (x_3 + \lambda_3 g) [(x_3 + \lambda_3 g)^{1/2}, \quad x_2(t_o) = x_{20} \\
x_3 &= g (x_3 + \lambda_3 g) [(x_3 + \lambda_3 g)^{1/2}, \quad x_3(t_o) = x_{30} \\
x_4 &= h(x_i, x_j)H(h), \quad x_4(t_o) = 0 \\
\dot{\lambda}_1 &= -2 \lambda_2 h(x_i, x_j)H(h), \quad \lambda_1(t_f) = -1 \\
\dot{\lambda}_2 &= 2 \lambda_1 h(x_i, x_j)H(h), \quad \lambda_2(t_f) = 0 \\
\dot{\lambda}_3 &= -\lambda_3 [(x_3 + \lambda_3 g)^{1/2}, \quad \lambda_3(t_f) = 0 \\
\dot{\lambda}_4 &= 0, \quad x_4(t_f) = 0
\end{align*}
\]

The solution of this set of nonlinear differential equations with the associated boundary conditions establishes the optimal trajectory and optimal control. Needless to say, this will not be an easy task. We shall examine this problem again, in Section 13.3-2, and determine a numerical solution for this optimization problem with a state variable inequality constraint.

4.4 Hamilton-Jacobi equation and continuous dynamic programming

Let us consider once more the problem of minimizing

\[
J = \int_{t_o}^{t_f} \phi(x(t), u(t), t) \, dt
\]

subject to the equality constraints

\[
x = \Gamma(x(t), u(t), t), \quad x(t_o) = x_o
\]

and the control variable inequality constraint

\[
u(t) \in \mathcal{U}
\]

where \( \mathcal{U} \) is a possibly infinite or semi-infinite closed interval, the admissible input set, which may depend on \( x(t) \) and \( t \). Let us further assume, for the moment, that \( t_f \) is fixed and \( x(t_f) \) is unspecified. Suppose that we have calculated \( \mathbf{u}(t) \) and \( \mathbf{x}(t) \) to be the optimal control and trajectory. The cost function is then a function of the initial state, \( x(t_o) \), and the initial time, \( t_o \), only. It is convenient to give this a special symbol such as

\[
V(x_o, t_o) \doteq J(\mathbf{x}, \mathbf{u}) = \int_{t_o}^{t_f} \phi(\mathbf{x}(t), \mathbf{u}(t), t) \, dt
\]

so that \( V(x_o, t_o) \) is the minimum value of the performance index when the initial system state is \( x_o \) and the initial time is \( t_o \). \( V(x_o, t_o) \) is a function only of \( x_o \) and \( t_o \) since \( \mathbf{x}(t) \) and \( \mathbf{u}(t) \) are known (optimal) values for all \( t \in [t_o, t_f] \).
We now consider a time $\Delta t$ between $t_0$ and $t_f$ and rewrite the cost function, Eq. (4.4-4), as
\[
V(x_0, t_0) = \int_{t_0}^{t_f} \phi(x, \bar{u}, t) dt + \int_{t_0}^{t_f} \phi(x, \bar{u}, t) dt
\]
\[
= \int_{t_0}^{t_f} \phi(x, \bar{u}, t) dt
\]
\[
J_i(\bar{x}, \bar{u}) = J_i(\bar{x}, \bar{u})
\]

If we now assume that $\phi$ is smooth over the interval $t_0$ to $t_f + \Delta t$ and that $\Delta t$ is sufficiently small, we may rewrite the $J_i$ term as
\[
J_i = \Delta \phi[\bar{x}(t_0 + \alpha \Delta t), \bar{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t], \quad 0 < \alpha < 1
\]
\[
(4.4-6)
\]

The second part of the cost function is simply
\[
V_2 = V[\bar{x}(t_0 + \Delta t), t_0 + \Delta t] = \int_{t_0 + \Delta t}^{t_f} \phi[\bar{x}(t), \bar{u}(t), t] dt
\]
\[
(4.4-7)
\]

This is because of the fundamental theorem of dynamic programming which asserts that any part of an optimal trajectory is an optimal trajectory. Therefore, we have
\[
J_i = \Delta \phi[\bar{x}(t_0 + \alpha \Delta t), \bar{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t]
\]
\[
(4.4-8)
\]

But this contradicts the assumption that $\bar{u}(t)$ is an optimal control. However, by the definition of $V$, $t_f \geq V(t)$; thus $J_2 = V(t)$. We will now write the cost function along the optimal trajectory as
\[
V(x_0, t_0) = \Delta \phi[\bar{x}(t_0 + \alpha \Delta t), \bar{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t]
\]
\[
+ \int_{t_0 + \Delta t}^{t_f} \phi[\bar{x}(t), \bar{u}(t), t] dt
\]
\[
(4.4-9)
\]

By expanding the last term in this equation in a Taylor's series about $\Delta t = 0$, we have
\[
V(x_0, t_0) = \Delta \phi[\bar{x}(t_0 + \alpha \Delta t), \bar{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t]
\]
\[
+ \int_{t_0 + \Delta t}^{t_f} \phi[\bar{x}(t_0 + \alpha \Delta t), \bar{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t]
\]
\[
+ \Delta t \bar{x}(t_0 + \Delta t) + \cdots
\]
\[
(4.4-10)
\]

Upon taking the limit as $\Delta t$ approaches zero and recalling the equality constraint of Eq. (4.4-2), we have, finally, the Hamilton-Jacobi equation
\[
\frac{\partial V(x_0, t_0)}{\partial t} + \phi(\bar{x}(t_0), \bar{u}(t_0), t_0) + \left[\frac{\partial V(x_0, t_0)}{\partial x_0}\right]^T f(\bar{x}(t_0), \bar{u}(t_0), t_0) = 0
\]
\[
(4.4-11)
\]

In this expression, we see that if we define
\[
\lambda(t_0) = \frac{\partial V(x_0, t_0)}{\partial x_0}
\]
\[
(4.4-12)
\]

we may then rewrite the Hamilton-Jacobi equation, dropping the subscript "o" for convenience, as
\[
\frac{\partial V(x, t)}{\partial t} + H(x, \bar{u}, x, t) = 0
\]
\[
(4.4-13)
\]

It is important for us to stress here that this Hamiltonian is the Hamiltonian evaluated (at time $t_0$) for the optimum control $\bar{u}(t)$, since we have been assuming all along that $\phi$ was evaluated about the optimal control and state. Thus, yet another way for us to write the Hamilton-Jacobi equation is
\[
\frac{\partial V(x, t)}{\partial t} = -H(x, \frac{\partial V}{\partial x}, t)
\]
\[
(4.4-14)
\]

where
\[
H(x, \frac{\partial V}{\partial x}, t) = \text{Min}_{u(t)} H(x(t), u(t), x(t) = \frac{\partial V(x, t)}{\partial x}, t)
\]
\[
(4.4-15)
\]

When $t_f$ is fixed and $x(t_f)$ is unspecified, it is an easy matter for us to show from Eq. (4.4-4) that the initial condition for the Hamilton-Jacobi equation is
\[
V[x(t_f), t_f] = 0
\]
\[
(4.4-16)
\]

If we had obtained the Hamilton-Jacobi equation for the cost function
\[
J = \theta[x(t_f), t_f] + \int_{t_0}^{t_f} \phi[x(t), u(t), t] dt
\]
\[
(4.4-17)
\]

we would have obtained the same Hamilton-Jacobi equation (4.4-13) with the initial condition (at the terminal time)
\[
V[x(t_f), t_f] = \theta[x(t_f), t_f]
\]
\[
(4.4-18)
\]

Needless to say, the Hamilton-Jacobi equation cannot be easily solved in general. However, when it can, $u(t)$ is determined as a function of $x(t)$, or in other words, we find a feedback control law which is highly desirable. The Hamilton-Jacobi partial differential equation is equivalent to the functional equation of dynamic programming or Bellman's equation [11,12,13]. It is sometimes called the Hamilton-Jacobi-Bellman equation [14].

**Example 4.4-1**

Let us consider the linear constant differential system described by
\[
\dot{x} = Ax(t) + bu(t), \quad x(0) = x_0
\]

where $A$ is an $n \times n$ matrix and $b$ is an $n$ vector. Any $u(t)$ is assumed to be admissible. We wish to find $u(t)$ as a function of $x(t)$ such that
\[
J = \frac{1}{2} \int_0^t [x^2 Q x + r u^2] dt
\]
We now consider a time $\Delta t$ between $t_0$ and $t_1$ and rewrite the cost function, Eq. (4.4.4), as
\[
V(x_0, t_0) = \int_{t_0}^{t_1} \phi(x, u, t) dt + \int_{t_1}^{t_2} \phi(x, u, t) dt
\]
\[
= J_f(x(f), \hat{u}(f))
\]
If we now assume that $\phi$ is smooth over the interval $t_0$ to $t_0 + \Delta t$ and that $\Delta t$ is sufficiently small, we may rewrite the $J_t$ term as
\[
J_1 = \Delta t \phi(x(t_0 + \alpha \Delta t), \hat{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t), \quad 0 < \alpha < 1
\]
The second part of the cost function is simply
\[
V_2 = V[x(t_0 + \Delta t), t_0 + \Delta t] = \int_{t_0 + \Delta t}^{t_1} \phi(x(t), \hat{u}(t), t) dt
\]
This is so because of the fundamental theorem of dynamic programming which asserts that any part of an optimal trajectory is an optimal trajectory. 

To show that $J_1$ is $V[x(t_0 + \Delta t), t_0 + \Delta t]$, we observe that the value of $J_1$ depends only on the state $x(t_0 + \Delta t)$ and the control $\hat{u}(t)$ in the time interval from $t_0 + \Delta t$ to $t_f$. If $J_1$ was greater than $V_2$, then there must have existed a control such that
\[
J_1(x, \hat{u}) + \int_{t_0 + \Delta t}^{t_1} \phi(x(t), \hat{u}(t), t) dt > V(x_0, t_0)
\]
But this contradicts the assumption that $\hat{u}$ is an optimal control. However, by the definition of $V_2$, $J_1 \geq V_2$; thus $J_1 = V_2$.

We will now write the cost function along the optimal trajectory as
\[
V(x_0, t_0) = \Delta t \phi(x(t_0 + \alpha \Delta t), \hat{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t) + V[x(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t]
\]
By expanding the last term in this equation in a Taylor’s series about $\Delta t = 0$, we have
\[
V(x_0, t_0) \approx \Delta t \phi(x(t_0 + \alpha \Delta t), \hat{u}(t_0 + \alpha \Delta t), t_0 + \alpha \Delta t) + V[x(t_0), t_0]
\]
\[
\left[\frac{\partial V(x(t_0), t_0)}{\partial x} \right] \Delta t + \left[\frac{\partial V(x(t_0), t_0)}{\partial x} \right] \Delta t + \left[\frac{\partial^2 V(x(t_0), t_0)}{\partial x^2} \right] \Delta t \cdot \Delta t + \cdots
\]
Upon taking the limit as $\Delta t$ approaches zero and recalling the equality constraint of Eq. (4.4.2), we have, finally, the Hamilton-Jacobi equation
\[
\frac{\partial V(x(t_0), t_0)}{\partial t} + \phi(x(t_0), \hat{u}(t_0), t_0) + \left[\frac{\partial V(x(t_0), t_0)}{\partial x} \right] \hat{x} + \left[\frac{\partial^2 V(x(t_0), t_0)}{\partial x^2} \right] \hat{x} + \cdots = 0
\]
In this expression, we see that if we define
\[
\lambda(t_0) = \frac{\partial V(x(t_0), t_0)}{\partial x}
\]
we may then rewrite the Hamilton-Jacobi equation, dropping the subscript “o” for convenience, as
\[
\frac{\partial V(x(t), t)}{\partial t} + H(x, \lambda, t) = 0
\]
It is important for us to stress here that this Hamiltonian is the Hamiltonian evaluated at time $t_f$ for the optimum control $\hat{u}(t)$, since we have been assuming all along that $\phi$ was evaluated about the optimal control and state. Thus, yet another way for us to write the Hamilton-Jacobi equation is
\[
\left[\frac{\partial V(x(t), t)}{\partial x} \right] = -H(x, \frac{\partial V(x(t), t)}{\partial x}, t)
\]
where
\[
H(x, \frac{\partial V(x(t), t)}{\partial x}, t) = \min_{u(t)} \left\{ V(x(t), u(t), t) = \frac{\partial V(x(t), t)}{\partial x}, t \right\}
\]
When $t_f$ is fixed and $x(t_f)$ is unspecified, it is an easy matter for us to show from Eq. (4.4.4) that the initial condition for the Hamilton-Jacobi equation is
\[
V[x(t_f), t_f] = 0
\]
If we had obtained the Hamilton-Jacobi equation for the cost function
\[
J = \theta[x(t_f), t_f] + \int_{t_0}^{t_f} \phi[x(t), u(t), t] dt
\]
we would have obtained the same Hamilton-Jacobi equation (4.4.13) with the initial condition (at the terminal time)
\[
V[x(t_f), t_f] = \theta[x(t_f), t_f]
\]
We now consider the linear constant differential system described by
\[
\dot{x} = Ax(t) + bu(t), \quad x(0) = x_0
\]
where $A$ is an $n \times n$ matrix and $b$ is an $n$ vector. Any $u(t)$ is assumed to be admissible. We wish to find $u(t)$ as a function of $x(t)$ such that
\[
J = \frac{1}{2} \int_0^{t_f} [\dot{x}^T Q x + r u^2] dt
\]
is a minimur; \( r \) is a positive constant semidefinite matrix, and \( r \) is positive.

The Hamiltonian for the problem is

\[
H(x, u, \lambda, t) = \frac{1}{2} x^T Q x + \frac{1}{2} r u^2 + \lambda^T A x + \lambda^T b u
\]

We need to find the control \( u \) which minimizes the Hamiltonian. This is

\[
\frac{\partial H}{\partial u} = 0 = ru + b^T \lambda
\]

so

\[
u = -b^T \lambda r^{-1}
\]

and the Hamiltonian becomes

\[
H(x, \lambda, t) = \frac{1}{2} x^T Q x + \lambda^T A x - \frac{1}{2} \lambda^T b b^T \lambda r^{-1}
\]

Since the system and the \( Q \) and \( r \) terms are time invariant and since the optimization is for a process of infinite duration, it follows that \( V(x, t) \) will depend only upon the initial state \( x \). This implies that

\[
\frac{\partial V(x, t)}{\partial t} = 0
\]

Therefore, since \( \lambda = \frac{\partial V}{\partial x} \), the Hamilton-Jacobi equation becomes

\[
\frac{1}{2} x^T Q x + \left( \frac{\partial V}{\partial x} \right)^T A x - \frac{1}{2} \left( \frac{\partial V}{\partial x} \right)^T b b^T \lambda r^{-1} = 0
\]

If we assume a solution

\[
V(x, t) = \frac{1}{2} x^T P x
\]

we see that

\[
\frac{\partial V}{\partial x} = P x
\]

and the Hamilton-Jacobi equation may be written as

\[
x^T \left[ \frac{1}{2} Q + \frac{1}{2} P A + \frac{1}{2} A^T P - \frac{1}{2} P b b^T P r^{-1} \right] x = 0
\]

which says that, for any nonzero \( x(t) \), the matrix \( P \) must satisfy the \( n(n+1)/2 \) algebraic equations (the \( P \) matrix is symmetric)

\[
Q + P A + A^T P - P b b^T P r^{-1} = 0
\]

This equation is solved for \( P \), and then the control is computed from

\[
u = -b^T \lambda r^{-1} = -b^T \left( \frac{\partial V}{\partial x} \right) = -b^T P x r^{-1}
\]

If we further consider the system

\[
\dot{x}_1 = x_2, \quad x_1(0) = x_{10}
\]

\[
\dot{x}_2 = u, \quad x_2(0) = x_{20}
\]

and the cost function

\[
J = \frac{1}{2} \int_0^\infty (4x_1^2 + u^2) \, dt
\]

it is easy for us to show that the optimum control is given by

\[
u = -2x_1 - 2x_2
\]

Example 4.4-2

Consider the system

\[
\dot{x} = -x^3 + u, \quad x(0) = x_0
\]

with cost function

\[
J = \frac{1}{2} \int_0^\infty (x_1^2 + u^2) \, dt
\]

where it is desired to determine the optimal feedback control. We accomplish this by forming the Hamiltonian

\[
H(x, u, \lambda, t) = \frac{1}{2} x^T x + \frac{1}{2} u^2 + \lambda x - \lambda^T b u
\]

We then set \( \frac{\partial H}{\partial u} = 0 \) and note that \( \lambda = \frac{\partial V}{\partial x} \) to obtain \( u = -\lambda \); then

\[
H(x, \frac{\partial V}{\partial x}) = \frac{1}{2} x^T x - \frac{1}{2} \left[ \frac{\partial^2 V(x, t)}{\partial x^2} \right] x^3 + \frac{1}{4} x^4 = 0
\]

The Hamilton-Jacobi equation is

\[
\frac{\partial V(x, t)}{\partial t} - \frac{1}{2} \left[ \frac{\partial^2 V(x, t)}{\partial x^2} \right] x^3 + \frac{1}{4} x^4 = 0
\]

with \( V(x(t), t) = 0 \).

If the optimization interval is infinite, then \( \frac{\partial V}{\partial t} = 0 \), and we need to solve the differential equation

\[
\frac{dv(x)}{dx} x^3 - v x^3 = 0
\]

with \( V(0) = 0 \) as the initial condition. We may approximate the solution to this ordinary differential equation by a series expansion

\[
V(x) = p_0 + p_1 x + \frac{1}{2} p_2 x^2 + \frac{1}{6} p_4 x^3 + \frac{1}{4} p_5 x^4 + \cdots
\]

If we terminate the series after the fourth-order term, substitute the assumed solution into the differential equation, and equate like powers of \( x \) (up to \( x^4 \)), we obtain \( p_0 = p_1 = p_2 = 0, p_4 = 1, p_5 = -6 \). Thus the approximate closed-loop control is

\[
u = -\lambda = -\frac{dv}{dx} = -x + x^3
\]

We naturally may question the stability of the approximate control. However, with \( u \) as obtained, the system differential equation becomes

\[
\dot{x} = -x^3 + u = -x
\]

which is certainly stable.

A similar procedure to this could have been used to obtain an approximate solution to the nonlinear partial differential equation that is the Hamilton-Jacobi equation for this example. In this case, the \( p \)'s would be functions of time, and we would obtain matrix Riccati-type equations [15]. This approach has many attractive features. In particular, only initial condition problems need be solved. However, there are two disadvantages: There is no assurance of system stability; the number of matrix Riccati differential equations which must be solved
Example maximum principle. We shall illustrate these difficulties with a simple example.

If an expansion in \( x \) to order \( n \) is used for the approximate solution to \( V(x, t) \), the number of distinct Riccati-type differential equations increases greatly with the order of the differential system and the order of the polynomial. For the approximative solution to \( V(x, t) \), if an expansion in \( x \) to the \( N \) order is used for an \( n \) vector differential system, the number of distinct Riccati-type differential equations is

\[
E = \frac{n!}{(n-1)!} + f + \frac{n!}{(n-1)!} \frac{1}{(n-1)!} + \cdots
\]

for an assumed solution of the form

\[
V(x, t) = \sum_{j=1}^{n!} P_j x_j + \frac{1}{2} \sum_{j=1}^{n!} \sum_{k=1}^{n!} P_{jk} x_j x_k + \frac{1}{3} \sum_{j=1}^{n!} \sum_{k=1}^{n!} \sum_{l=1}^{n!} P_{jkl} x_j x_k x_l + \cdots
\]

If, for example, the solution to a four-vector differential system is approximated by terms up to and including the fourth power in \( x \) we need to solve sixty-nine differential equations to obtain the closed-loop control.

Our discussion of the second variation technique, the invariant imbedding procedure, and specific optimal control using the quasilinearization approach will point out many interesting interconnections with the approach used to obtain the solution to this example.

In our development thus far, we have assumed that the terminal time, \( t_f \), is fixed. It is possible to remove this restriction with the result that the Hamilton-Jacobi equation (4.4-13), (4.4-14), or (4.4-15) is still applicable. The initial condition for the Hamilton-Jacobi equation is still Eq. (4.4-18) and, in addition, the terminal time is determined by the relation

\[
H(x, \frac{\partial V}{\partial x}, t) + 1 = 0, \quad \text{at} \quad t = t_f
\]

(4.4-19)

which holds if the problem is a minimum time problem such that

\[
V(x, t) = t_f - t
\]

(4.4-20)

If, further, the differential system is time invariant, the Hamiltonian is equal to \(-1\) at all times along the optimal trajectory.

We may formally obtain the Pontryagin maximum principle by taking appropriate partial derivatives of the Hamilton-Jacobi equations (Problem 9). However, the resulting maximum principle is not applicable to as broad a class of problems as is possible. The reason for this is that it is necessary that \( V(x, t) \) be smooth or, in other words, twice continuously differentiable with respect to \( x \) in order to obtain the Hamiltonian canonical equations of the maximum principle. We shall illustrate these difficulties with a simple example.

Example 4.4-3

A second-order example will now be discussed to illustrate that the assumption of the differentiability of \( V(x, t) \) does not hold in some of the simplest cases.

We will consider the problem of transferring the system represented by the differential equations

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u
\]

from an initial state \( x_0 \) to the origin in minimum time. We assume that the admissible set for the scalar control is described by \( |u| \leq 1 \).

This problem can be solved by the Pontryagin maximum principle. In the time optimal problem

\[
J = \int_{t_0}^{t_f} (1) \, dt
\]

Therefore, the Hamiltonian is

\[
H(x, u, \lambda_1, t) = 1 + \lambda_1 x_2 + \lambda_2 u
\]

The adjoint equations are

\[
\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1
\]

The solutions to these equations are

\[
\lambda_1 = C_1, \quad \lambda_2 = C_2 - C_1 t
\]

where \( C_1 \) is the initial condition on \( \lambda_1 \). The control which minimizes the Hamiltonian subject to \(|u| \leq 1\) is

\[
u = -\text{sign} \lambda_2 = \text{sign} (C_2 - C_1 t)
\]

The initial conditions \( C_1 \) and \( C_2 \) are not arbitrary but must be such that \( x(t_f) = 0 \) since it is desired to transfer the system \( x_0 \) to the origin in minimum time. When \( u = +1 \), the solution to the differential system equation is

\[
x_1 = t + x_2(0)
\]

\[
x_2 = \frac{t}{2} + x_2(0)t + x_1(0)
\]

If \( t \) is eliminated from the foregoing, we obtain

\[
x_1 = \frac{x_2^2}{2} + x_2(0) - \frac{x_2^2}{2}
\]

When \( u = -1 \), the solution to the differential system equations is

\[
x_1 = -t + x_2(0)
\]

\[
x_2 = \frac{-t^2}{2} + x_2(0)t + x_1(0)
\]

and if \( t \) is eliminated in the foregoing, we obtain

\[
x_1 = \frac{-x_2^2}{2} + x_2(0) + \frac{x_2^2}{2}
\]

By determining the constants \( C_1 \) and \( C_2 \) in terms of \( x_1 \) and \( x_2 \), it is a straightforward task for us to show that the control law is

\[
u = -\text{sign}[x_1(t) + 2x_2(t)|x_1(t)|]
\]

These equations represent the optimal control and trajectories for \( u = -1 \) and \( u = +1 \), respectively, and they indicate that these trajectories are segments of parabolas. Figure 4.4-1 is a plot of some of these parabolas.

The segment of the parabola which is not an optimal trajectory has been represented by a broken line. The optimal control can be determined from Fig. 4.4-1 and a knowledge of the state of the system.
sents the switching curve. When \( x \) lies below \( AOB \), \( u = +1 \) until the system state reaches the curve \( AO \), at which time the control switches to \(-1\). If \( x \) lies above \( AOB \), \( u = -1 \) until it reaches \( BO \), where it switches to \(+1\).

The optimal transition time \( T(x) \), which is the cost function \( J \) or \( V(x, t) \), can be determined from the solutions for \( x_1 \) and \( x_2 \). Figure 4.4-2 is a plot of \( T(x) \), the minimum time to transfer to the origin for the case in which the initial \( x_2 \) is held constant \((x_{20} = -2)\), and \( x_{10} \) is varied about the switching line.

From the graph it can be seen that \( \partial T(x)/\partial x \) has a discontinuity at the switching curve. It can be shown analytically that \( \partial T(x)/\partial x \) “flows up” as \( x_1 \) approaches \(+2\) from the left. Hence the Hamilton-Jacobi equation would not be applicable in examples of this type. This example indicates the loss of generality which results from deriving the maximum principle from the Hamilton-Jacobi-Bellman equations.

REFERENCES

PROBLEMS

1. Find the TPBVP which, when solved, yields the control, $u(t)$, and trajectory, $x(t)$, which minimize

$$J = \frac{1}{2} \int_0^1 (x^3 + u^2) \, dt$$

for the system

$$\dot{x} = -x^3 + u, \quad x(0) = 1$$

2. Find the control and trajectory which transfers the system

$$\dot{x}_1 = x_1, \quad x_1(0) = 0$$
$$\dot{x}_2 = u, \quad x_2(0) = 0$$

to the line

$$x_1(t) + x_2(t) = 1$$

such that

$$J = \frac{1}{2} \int_0^1 u(t) \, dt$$

is minimized.

3. Find the control and trajectory which transfers the system

$$\dot{x} = -x + u$$

from $x(0) = 10$ to $x(1) = 0$ such that

$$J = \frac{1}{2} \int_0^1 (x^3 + u^2) \, dt$$

is minimized.

4. Find the control and trajectory which minimizes

$$J = \frac{1}{2} \int_0^1 x^2(t) \, dt$$

subject to the inequality constraint $|u(t)| \leq 1$ for the system $\dot{x} = u$ such that $x(0) = 1, x(1) = 1$.

5. Determine the Weierstrass-Erdmann corner conditions for the minimization of the cost function

$$J = \int_0^1 x(2 - x)^2 \, dt$$

6. What is the Weierstrass $E$ function for the cost function of Problem 5?

7. For the system

$$\dot{x}_1 = x_1, \quad x_1(0) = 10$$
$$\dot{x}_2 = u, \quad x_2(0) = 0$$

find the control and trajectory which minimizes

$$J = t_f^2 + \frac{1}{2} \int_0^1 u^2 \, dt$$

if the desired final state is:

(a) $x_1(t_f) = x_2(t_f) = 0$. 
(b) $x_1(t_f) = 0, x_2(t_f) = \text{unspecified}.

8. Develop a second- and fourth-order approximation to the solution of the Hamilton-Jacobi equation to find the closed-loop control which minimizes

$$J = \frac{1}{2} \int_0^1 (x_1^2 + u^2) \, dt + \frac{1}{2} \int_0^1 \frac{\partial V}{\partial x} \, dx$$

for the system

$$\dot{x}_1 = x_2 + x_3^2 + \frac{\partial V}{\partial x_1}, \quad \dot{x}_2 = x_1 - x_2 + u$$

Compute and compare the actual numerical results when $t_f$ is infinite.

9. Derive the Pontryagin maximum principle from the Hamilton-Jacobi equation by calculating $(d/dt)(\partial V/\partial x)$ and $\partial V/\partial x$ as outlined in Section 4.4. Observe the differentiability requirement on $V(x, t)$.

10. Find the control vector which minimizes

$$J = \frac{1}{2} \int_0^1 (x_1^2 + u^2 + u_2^2) \, dt$$

for the system described by

$$\dot{x}_1 = u_1 + u_2, \quad \dot{x}_2 = x_1(t_1) \quad x(0) = 1$$

Use the maximum principle and the Hamilton-Jacobi equations to find the optimum control vector.

11. Set up the differential equations and boundary conditions to minimize for $t_f$ unspecified

$$J = \int_0^1 u^2 dt + t_f x_1(t_f)$$

subject to the constraints

a) $\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = u$

b) $x(0) = 0$

c) $|u| \leq 1; |x_1| \leq 10$

d) $x_1(t_f) = t_f, x(t_f) = x_1(t_f)$

12. Set up the equations and boundary conditions to optimize the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

for the performance index with $t_f$ unspecified

$$J = \int_0^1 x^2 dt + t_f x_1(t_f)$$

subject to all of the following constraints

a) $x^2(0) = [1, 0, 0]$

b) $x_1(t_f) = x_2(t_f)$

c) $x_3(t_f) = 0$

d) $|u| \leq 1$

e) $\int_0^1 u^2 dt = 1$

13. Find the Hamilton-Jacobi equation for the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - x_1 + u$$
if the performance index is
\[ J = \int_0^T (x^2 + u^2) \, dt \]

14. Show that the solution of the Hamilton-Jacobi equation for the system
\[ \dot{x} = Ax + u, \quad A^T A + A = 0, \quad \|u\| \leq 1 \]
and the cost function
\[ J = \int_0^T dt = T \]
is
\[ V(x) = \|x\| \]
What is the optimal control?

15. Find the optimal control to minimize
\[ J = \int_0^T dt \]
for the system
\[ \dot{x} = -x + u, \]
when
\[ x(0) = 1, \quad x(T) = 0 \]
\[ \|u\| \leq 1 + |x| \]

In this chapter, we will illustrate some, but certainly by no means all, or even a majority, of the optimal control problems for which closed-form analytic solutions have been obtained. The problems we will solve in this chapter are very important in their own right and illustrate the use of the maximum principle for problems in which closed-form analytic solutions may be obtained. Specifically, we will discuss the linear regulator problem, the first solution of which was due to Kalman [1, 2, 3, 4]. We then discuss the minimum time problem which has been considered by Pontryagin [5], Bellman [6], LaSalle [7], and many others [8 through 13].

A characteristic of some minimum time problems is the possibility of a singular solution. The possibility of singular solutions is well-recognized in the variational calculus literature and has been extensively discussed for control problems by Johnson [14, 15, 16] and others. Minimum fuel problems for linear differential systems are then discussed. A variety of authors, but notably Athans, have discussed various aspects of minimum fuel problems including the possibility of singular solutions [17 through 20]. Finally, the minimum time, minimum fuel, and minimum energy control of self-adjoint systems are discussed. It is certainly true that the self-adjoint assumption, coupled with the need for as many control inputs as state variables, seriously restricts the practical usefulness of the solutions, particularly for high-order systems. However, the relative ease with which the control can be computed makes this an excellent example for a relatively thorough analysis.
Many other minimal control problems are solved in this book other than the ones in this chapter. Discrete and distributed parameter problems are reserved for the next two chapters. Chapter 11 discusses several optimal control problems with regard to observability and controllability. Nonlinear problems, which include the majority of optimal control problems, are discussed in Chapters 13, 14, and 15. The literature in this area is very extensive. For an excellent survey of many other problems plus a lengthy bibliography, we refer to the survey papers of Paiwonsky [22] and Athans [23].

5.1 The linear regulator

We will now study a particular control problem which has as its solution a linear feedback control law. It occurs where we have a linear differential system

\[ \dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0 \]  

and wish to find the control which minimizes the cost function (for \( t_f \) fixed)

\[ J = \int_{t_0}^{t_f} \left[ x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right] dt \]  

Clearly, there is no loss of generality in assuming \( Q, R, \) and \( S \) to be symmetric. We may obtain the solution to this problem via the maximum principle or the Hamilton-Jacobi equation. Here, we will use the former method. The Hamiltonian is

\[ H[x(t), u(t), \lambda(t), t] = \frac{1}{2} x^T(t)Q(t)x(t) + \frac{1}{2} u^T(t)R(t)u(t) + \lambda^T(t)Ax(t) + \lambda^T(t)Bu(t) \]  

Application of the maximum principle requires that, for an optimum control,

\[ \frac{\partial H}{\partial u} = 0 = R(t)u(t) + B(t)^T\lambda(t) \]  

and

\[ \frac{\partial H}{\partial x} = -\lambda = Q(t)x(t) + A^T(t)\lambda(t) \]  

with the terminal condition

\[ \lambda(t_f) = \frac{\partial J}{\partial x(t_f)} = Sx(t_f) \]  

Thus we require that

\[ u(t) = -R^{-1}(t)B(t)^T\lambda(t) \]  

and we shall inquire whether we may convert this to a closed-loop control by assuming that the solution for the adjoint is similar to Eq. (5.1-6)

\[ \lambda(t) = P(t)x(t) \]  

If we substitute this relation into Eqs. (5.1-1) and (5.1-7), we see that we must require

\[ \dot{x} = A(t)x(t) - B(t)R^{-1}(t)B^T(t)P(t)x(t) \]  

Also, from Eqs. (5.1-8) and (5.1-5) we require

\[ \dot{\lambda} = \dot{P}(t)x(t) + P(t)\lambda(t) = -A^T(t)P(t)x(t) - \lambda^T(t)Q(t)x(t) \]  

By combining Eqs. (5.1-9) and (5.1-10) we have

\[ \dot{P} + P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t)x(t) = 0 \]  

Since this must hold for all nonzero \( x(t) \), the term premultiplying \( x(t) \) must be zero. Thus the \( P \) matrix, which we see is an \( n \times n \) symmetric matrix and which has \( n(n+1)/2 \) different terms, must satisfy the matrix Riccati equation — which, as we shall see later, must be positive definite —

\[ \dot{P} = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) - Q(t) \]  

with a terminal condition given by Eqs. (5.1-6) and (5.1-8)

\[ P(t_f) = S \]  

Thus we may solve the matrix Riccati equation backward in time from \( t_f \) to \( t_0 \), store the matrix

\[ u(t) = -K(t)x(t) \]  

It is important to note that all components of the state vector must be accessible. We will remove this restriction in Chapter 11 when we discuss the ideal observer. A block diagram for accomplishing this solution to the regulator problem is shown in Fig. 5.1-1. If we compute the second variation, we find that

\[ \delta^2 J = \frac{1}{2} \delta x^T(t_f)S \delta x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[ \delta x^T(t)Q(t)\delta x(t) + \delta u^T(t)R(t)\delta u(t) \right] dt \]  

Fig. 5.1-1 Optimum linear closed-loop regulator.
Thus, $Q$ and $S$ must be at least positive semidefinite in order to establish the sufficient condition for a minimum. In addition, we know from Eq. (5.1-7) that $R$ must have an inverse; therefore, it is sufficient that $R$ be positive definite and that $Q$ and $S$ be at least positive semidefinite.

In some cases it may turn out that certain elements of the $S$ matrix are large enough to give computational difficulties. In this case, it is possible and perhaps desirable to obtain an inverse Riccati differential equation; we let

$$P(t)P^{-1}(t) = I \quad (5.1-17)$$

and, by differentiating, we obtain

$$\dot{P}P^{-1}(t) + P(t)\dot{P}^{-1} = 0 \quad (5.1-18)$$

such that we obtain an "inverse" matrix Riccati equation

$$\dot{P}^{-1} = A(t)P^{-1}(t) + P^{-1}(t)A^T(t) - B(t)R^{-1}(t)B^T(t) + P^{-1}(t)Q(t)P^{-1}(t) \quad (5.1-19)$$

with

$$P^{-1}(t) = S^{-1} \quad (5.1-20)$$

In this way, for example, it is possible to solve the Riccati equation such that $S^{-1} = [0]$, the null matrix, which will require that each and every component of the state vector approach the origin as the time approaches the terminal time. The "gains" $K(t)$, or at least some components of them, become infinite at the terminal time in this case. It is also necessary to assume certain controllability requirements here, as we shall see in Chapter 11.

It is possible to write the nonlinear $n \times n$ matrix Riccati equation with a terminal condition as a $2n$ vector linear differential equation with two-point boundary conditions. We will use this approach, in part, to solve a Riccati equation associated with a filtering problem in Chapter 9. Our discussion of the second variation method in Chapter 13 will also make use of a Riccati transformation.

Example 5.1-1

Consider the scalar system

$$\dot{x} = -k x(t) + u(t), \quad x(t_0) = x_0$$

with the cost function

$$J = \frac{1}{2}x^2(t_1) + \frac{1}{2} \int_{t_0}^{t_1} [2x^2(t) + u^2(t)] dt$$

The Riccati equation, Eq. (5.1-12), becomes

$$\dot{p} = p + p^2 - 2, \quad p(t_1) = s$$

which has a solution that we may write as either

$$p(t) = -0.5 + 1.5 \tanh (-1.5t + \xi_1) \quad \checkmark$$

or

$$p(t) = -0.5 + 1.5 \coth (-1.5t + \xi_2) \quad \checkmark$$

where $\xi_1$ and $\xi_2$ are adjusted such that $p(t_1) = s$.

For example, if

(a) $s = 0$, $t_f = 1$, then $\xi_1 = 1.845$ radians, which gives

$$K(t) = -R^{-1}B^TP = 0.5 - 1.5 \tanh (-1.5t + 1.845)$$

Since $s = 0$, we are not particularly weighting the state at the final time, and the "gain" (and control) goes to zero at the final time.

(b) $s = 10$, $t_f = 10$, then $\xi_2 = 15.1425$ radians. In this case we are applying a great weight to the error at $t = t_f$, and the gain becomes large ($-10$) at the terminal time.

(c) $s = \infty$, the Riccati equation cannot be solved directly since it has an infinite initial condition. The inverse Riccati equation can be solved with zero terminal condition to give

$$K^{-1}(t) = \left[0.25 + 0.75 \tanh (-1.5t + 1.5t_f - 0.346)\right]$$

As $t_f$ becomes infinite, it is easy to show that $K(t)$ becomes unity and, as is expected, the feedback gain becomes constant. Figure 5.1-2 illustrates $K(t)$, the "Kalman gains" as they are sometimes called, for these three cases for this particular problem.

Example 5.1-2

Let us consider the optimum closed-loop control for a nuclear reactor system. Specifically, we wish to consider a very simple reactor model with zero temperature feedback. Only one group of delayed neutrons will be used.

The reactor kinetics are described by the equations

$$\dot{n} = \frac{(\rho - \beta)n + \lambda c}{\Lambda}, \quad \dot{c} = \frac{6n}{\Lambda} - \lambda c$$

where the neutron density, $n$, and the precursor concentration, $c$, are the state variables, and the reactivity $\rho$ is the control variable. The system has the initial
We will now develop a method of feedback control about the optimal trajectory which minimizes a cost function $J_2$; it will be quadratic in deviation from the nominal (optimal for $J_1$) trajectory and control.

Having formulated a model for the nuclear reactor system and determined the optimal trajectories, we now desire to determine the linearized system coefficient matrix about the optimal trajectory. The deviations of the state and control variables about the optimal or nominal trajectories are expressed by

$$n = n_0(t) + \Delta n(t), \quad c = c_0(t) + \Delta c(t)$$

$$\rho = \rho_0(t) + \Delta \rho(t), \quad u = u_0(t) + \Delta u(t)$$

where $u$ is the control variable. Chapter 14 on quasilinearization indicates how the nonlinear two-point boundary value problem resulting from the use of optimal control theory may be used to obtain the optimum control and trajectory, which are shown in Fig. 5.1-3, for the following system parameters

$$\lambda = 0.1 \text{ sec}^{-1} \quad n_0 = 10 \text{ kW}$$
$$\Lambda = 10^{-3} \text{ sec} \quad \beta = 0.0064$$
$$d = 5 \quad t_f = 0.5 \text{ sec}$$

We will now develop a method of feedback control about the optimal trajectory which minimizes a cost function $J_2$; it will be quadratic in deviation from the nominal (optimal for $J_1$) trajectory and control.

Having formulated a model for the nuclear reactor system and determined the optimal trajectories, we now desire to determine the linearized system coefficient matrix about the optimal trajectory. The deviations of the state and control variables about the optimal or nominal trajectories are expressed by

$$n = n_0(t) + \Delta n(t), \quad c = c_0(t) + \Delta c(t)$$

$$\rho = \rho_0(t) + \Delta \rho(t), \quad u = u_0(t) + \Delta u(t)$$

$$\dot{\lambda} = \frac{(\rho - \beta \lambda)}{\Lambda} + c$$
$$\dot{c} = \frac{\beta n}{\Lambda} - \lambda \rho$$

The control variable therefore becomes $\rho$, and $\rho$, in effect, thus becomes a state variable. The kinetics equations may then be rewritten as

$$\dot{n} = n_0(t) + \Delta n(t), \quad c = c_0(t) + \Delta c(t)$$

$$\rho = \rho_0(t) + \Delta \rho(t), \quad u = u_0(t) + \Delta u(t)$$

$$\dot{\lambda} = \frac{(\rho - \beta \lambda)}{\Lambda} + c$$
$$\dot{c} = \frac{\beta n}{\Lambda} - \lambda \rho$$

The control variable therefore becomes $\rho$, and $\rho$, in effect, thus becomes a state variable. The kinetics equations may then be rewritten as
The state vector 
\[ \Delta x^i(t) = [\Delta n(t), \Delta \phi(t), \Delta \rho(t)] \]

The linearized model becomes 
\[ \Delta \dot{x} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ \beta & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Delta u \\
= A(t)\Delta x(t) + b(t)\Delta u(t) \]

where 
\[ a_{11}(t) = \rho_n(t) - \beta \Lambda, \quad a_{12}(t) = \frac{\mu_n(t)}{\Lambda} \]

To complete our design of the closed-loop controller, we must evaluate \( A(t) \) and \( b(t) \) about the optimum or nominal trajectories, select the \( R, Q, \) and \( S \) matrices, and solve the associated Riccati equation. The nominal trajectory, control, and time-varying gains are then stored and used to complete the closed-loop controller design.

The choice of the \( R, Q, \) and \( S \) matrices to minimize 
\[ J_2 = \frac{1}{2} \Delta x^T(t) S \Delta x(t) + \frac{1}{2} \int_t^T [\Delta x^T(t) Q(t) \Delta x(t) + r(t) \Delta u^2(t)] \, dt \]
is somewhat arbitrary and can perhaps best be done here by experimentation. We can accomplish this only after we have obtained a knowledge of possible disturbances which may drive the system off of the nominal trajectory. Let us assume that we will use

\[ Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10^4 \end{bmatrix}, \quad S = 0, \quad r = 1 \]

In Chapter 13 the second variation and neighboring optimal methods of control-law computation will lead us to a method for choosing the proper weighting matrices for a variety of cases, in particular, for relating \( J_1 \) and \( J_2 \).

The control, \( \Delta u(t) \), is computed from 
\[ \Delta u(t) = -R^{-1}(t)B^TP(t)\Delta x(t) \]

where it is necessary to solve the \( 3 \times 3 \) matrix Riccati equation, having six different first-order differential equations, to obtain \( P(t) \). Figure (5.1-4) illustrates the Kalman gains 
\[ -K(t) = [p_{31}(t), p_{32}(t), p_{33}(t)] \]

for this example. Figure (5.1-5) indicates how the complete closed-loop controller is obtained. It is interesting to note that, in an actual physical problem, the precursor concentration is not measurable, and therefore we need to add an "observer" of this particular state variable. We also need to discuss many more aspects of this problem such as disturbances and parameter variations. We will postpone further consideration of these important questions until we establish some foundation in state and parameter estimation and optimal adaptive control. We have, in this example, illustrated how a basically nonlinear problem may be linearized, and a linear time-varying closed-loop controller obtained, if a nominal trajectory is known. Since this can be accomplished for a variety of problems, we see that the linear regulator problem is indeed an important one.
5.2 The Linear Servomechanism

The linear regulator problem considered in the preceding section can be generalized in several ways. We can assume that we desire to find the control in such a way as to cause the output to track or follow a desired output state, \( \psi(t) \). We may also assume that there is a forcing function (not the control) for the system differential equations. Therefore, we will consider the minimization of

\[
J = \frac{1}{2} \| \psi(t_f) - z(t_f) \|^2 + \frac{1}{2} \int_{t_0}^{t_f} \left[ \| \psi(t) - z(t) \|^2 + \| u(t) \|^2 \right] dt \quad (5.2-1)
\]

for the system which contains an input or plant noise vector \( w(t) \)

\[
\dot{x} = A(t)x(t) + B(t)u(t) + w(t), \quad x(t_0) = x_0 \quad (5.2-2)
\]

\[
x(t) = C(t)x(t) \quad (5.2-3)
\]

The requirements on the various matrices are the same as in the preceding section. We proceed in exactly the same fashion as for the regulator problem. The Hamiltonian is, from Eq. (4.3-34),

\[
H(x, u, \lambda, t) = \frac{1}{2} \| \psi(t) - C(t)x(t) \|^2 + \frac{1}{2} \| u(t) \|^2
\]

\[
+ \lambda^T(t)[A(t)x(t) + B(t)u(t) + w(t)] \quad (5.2-4)
\]

We employ the maximum principle and set \( \frac{\partial H}{\partial u} = 0 \) to obtain

\[
u(t) = -R^{-1}(t)B^T(t)[P(t)x(t) - \psi(t)] \quad (5.2-5)
\]

and

\[
\frac{\partial H}{\partial x} = \dot{x} = C^T(t)Q(t)[C(t)x(t) - \psi(t)] + \lambda^T(t)A(t)x(t) \quad (5.2-6)
\]

with the terminal condition

\[
\lambda(t_f) = C^T(t_f)S[C(t_f)x(t_f) - \psi(t_f)] \quad (5.2-7)
\]

In order to attempt to determine a closed-loop control, we assume

\[
\lambda(t) = P(t)x(t) - \xi(t) \quad (5.2-8)
\]

We substitute this relation into the canonic equations and determine the requirements for a solution. By a procedure analogous to that of the preceding section, we easily obtain the following requirements

\[
P = -P(t)A(t) - A^T(t)P(t)
\]

\[
+ P(t)B(t)R^{-1}(t)B^T(t)P(t) - C^T(t)Q(t)C(t) \quad (5.2-9)
\]

\[
P(t_f) = C^T(t_f)S\xi(t_f) \quad (5.2-10)
\]

and

\[
\dot{\xi} = -[A(t) - B(t)R^{-1}(t)B^T(t)P(t)]\xi + P(t)w(t) - C^T(t)Q(t)\psi(t) \quad (5.2-11)
\]

\[
\xi(t_f) = C^T(t_f)S\psi(t_f) \quad (5.2-12)
\]

Thus we see that the linear servomechanism problem (posed of two parts: a linear regulator part, plus a prefilter to determine the optimal driving function from the desired value, \( \psi(t) \), of the system output. The optimum control law is linear and is obtained from Eq. (5.2-5) as

\[
u(t) = -R^{-1}(t)B^T(t)[P(t)x(t) - \xi(t)] \quad (5.2-13)
\]

Unfortunately, the optimal control is, in practice, often computationally unrealizable because it involves \( \xi(t) \) which must be solved backward from \( t_f \) to \( t_0 \) and, therefore, requires a knowledge of \( \psi(t) \) and \( w(t) \) for all time \( t \in [t_0, t_f] \). This is quite often not known at the initial time \( t_0 \).

Example 5.2-1

Let us consider the minimization of the cost function

\[
J = \frac{1}{2} \int_{t_0}^{t_f} [x(t) - \eta] \cdot [x(t) - \eta] dt
\]

for the system described by

\[
\begin{align*}
x_1 &= x_2, \\
x_2 &= u, \\
x_1(0) &= x_1(0) = x_0
\end{align*}
\]

We first use Eqs. (5.2-9) and (5.2-10) to obtain the Riccati equation for this example

\[
\dot{P}_{11} = -P_{11} - 1, \quad P_{11}(t_f) = 0
\]

\[
\dot{P}_{12} = -P_{11} + P_{12}P_{22}, \quad P_{12}(t_f) = 0
\]

\[
\dot{P}_{21} = -P_{21} + P_{22}, \quad P_{22}(t_f) = 0
\]

If we allow \( t_f \) to become infinite, we obtain the solution \( P_{11} = P_{22} = \sqrt{2} \), \( P_{12} = 1 \). Thus we have for the closed-loop control

\[
u = -R^{-1}B^TP(t-x(t) - \xi(t) = -x_1 - \sqrt{2}x_2 + \xi_2
\]

where we must determine \( \xi \) by solving Eqs. (5.2-11) and (5.2-12) which become for this example

\[
\begin{align*}
\dot{\xi}_1 &= \xi_1 - \eta_1, \quad \xi_1(t_f) = 0 \\
\dot{\xi}_2 &= \xi_2 + \sqrt{2}\xi_2, \quad \dot{\xi}_2(t_f) = 0
\end{align*}
\]

If \( \eta_1 = \alpha \), a constant, for \( t \) greater than zero, we are justified in obtaining the equilibrium solution for the \( \xi \) equation if \( t_f = \infty \) by setting \( \dot{\xi} = 0 \) to obtain

\[
\dot{\xi}_2 = 0.707\xi_1 = \eta_1 = \alpha.
\]

If \( \eta_1 = 1 - e^{-t} \), we will then find by a simple limiting process that for \( t_f = \infty \),

\[
\dot{\xi}_2(t) = 1 + \frac{1}{2 + \sqrt{2}} e^{-t}, \quad t \geq 0
\]

We may realize this solution as shown in Figure (5.2-1).

We note that if \( w(t) = \eta(t) = 0 \), or for that matter, any vector constant in time, the servomechanism problem reduces to a regulator problem except that it is an "output" regulator problem rather than a "state" regulator problem because of the presence of the output matrix \( C(t) \). It is not necessary...
for the system to be controllable in order to find a solution to the regulator problem. The only exception to this is in the limiting cases where $S$ becomes infinite or where $r_f$ becomes infinite. It is, however, necessary that the system be observable in order for a solution to the output regulator problem to exist. We will expand considerably on these ideas when we consider controllability, observability, and the reachable zone problem in Chapter 11.

It is possible to give a frequency-domain interpretation to the regulator and servomechanism problem for the infinite time interval case for a constant system. We will present this method, due to Kalman, in Chapter 9 where the duality concept will allow us to treat both the estimation and the control problems.

### 5.3 Bang bang control and minimum time problems

Maximum effort control problems have become increasingly important in a variety of applications. It is natural that we ask under what circumstances optimal controls will always be maximum effort, or bang bang. To do this, we will restrict each component of the control vector, $u(t)$, to some bounded interval. Let us consider the nonlinear differential system where the control enters in a linear fashion

$$
\dot{x} = f[x(t), t] + G[x(t), t]u(t), \quad x(t_0) = x_0
$$

(5.3-1)

and assume a performance index which, likewise, contains only linear terms in the control variable, such that the Hamiltonian will also be linear in $u(t)$.

$$
J = \int_{t_0}^{t_f} [\phi[x(t), t] + h^r[x(t), t]u(t)] dt
$$

(5.3-3)

$$
H[x(t), u(t), \lambda(t), t] = \phi[x(t), t] + h^r[x(t), t]u(t) + \lambda^r(t)[f[x(t), t] + G[x(t), t]u(t)]
$$

(5.3-4)

Since the Hamiltonian is linear in the control vector, $u(t)$, minimization of the Hamiltonian with respect to $u(t)$ requires that

$$
u_i = \begin{cases} a_i & \text{if } [h^r[x(t), t] + \lambda^r(t)G[x(t), t]]_i > 0 \\ b_i & \text{if } [h^r[x(t), t] + \lambda^r(t)G[x(t), t]]_i < 0 \end{cases}
$$

(5.3-5)

Thus we see that when the control vector appears linearly in both the equation of motion of the differential system and the performance index, and if in addition each component of the control vector is bounded, the optimal control is bang bang. The only exception to this occurs in cases where

$$
h^r[x(t), t] + \lambda^r(t)G[x(t), t] = 0 \quad \forall t \in [t_0, t_f]
$$

(5.3-6)

for then the Hamiltonian is not a function of $u(t)$ and cannot be minimized with respect to $u(t)$. When Eq. (5.3-6) holds for more than isolated points in time, the optimization problem is said to possess a singular solution, a problem which we will discuss in detail in the next section. A singular solution is possible with respect to a particular control component, $u_i$, if the $i$th component of Eq. (5.3-6) is zero.

For this problem, the canonical equations are obtained as

$$
\dot{x} = \frac{\partial H}{\partial \lambda} = f[x(t), t] + G[x(t), t]u(t)
$$

(5.3-7)

$$
-\lambda = \frac{\partial H}{\partial x} = \frac{\partial [\phi[x(t), t] + h^r[x(t), t]u(t)]}{\partial x} + \frac{\partial [G[x(t), t]u(t)]^T}{\partial x} \lambda(t)
$$

(5.3-8)

where $u(t)$ is determined via Eq. (5.3-5). Since we have not specifically stated the end conditions, we have carried the general problem about as far as is possible. When we specify information concerning the desired states at the terminal time and the initial condition vector, we have, as before, a two-point boundary value problem with half of the conditions specified at the initial time and half at the terminal time. A possible method of solution of the canonical equations for this formulation consists of reversing time in the canonical equations. Starting at the determined or specified terminal vector, which often is the origin of the state vector, we integrate back from this point with a constant control until a switching point is obtained from Eq. (5.3-5). Since no terminal conditions are present for half of the state variables, the method is, of necessity, cut and try. Chapters 13, 14, and 15 provide more systematic methods for solving this type of two-point boundary value problem.

We shall now illustrate various solutions to a particular case which results in bang bang control—the minimum time problem for constant linear systems with a scalar input. In this problem, we desire to transfer an $n$ vector constant differential system

$$
\dot{x} = Ax(t) + bu(t), \quad x(t_0) = x_0
$$

(5.3-9)

to the origin, $x(t_f) = 0$, in minimum time, such that we have for the cost function

$$
J = \int_{t_0}^{t_f} dt = t_f - t_0
$$

(5.3-10)
CONTROLLABILITY AND OBSERVABILITY
—THE SEPARATION THEOREM

In our previous work with the regulator and servomechanism problems, we noted that there were certain requirements, in addition to the definiteness of certain matrices, which must exist in order for the problem to have a meaningful solution. In this chapter we wish to examine these requirements, which we have postponed until now so that we might explore them using optimum control and filtering theory.

First we will examine an intrinsic characterization of the manner in which the output of a system is constrained with respect to the ability to observe the system states. Then we will examine the dual requirement and find the characterization of the manner in which a system is constrained with respect to control of the system states or system outputs. We will consider these requirements for both continuous and discrete systems and will thus prove the observability and controllability requirements for linear systems. Original efforts in this area are due to Kalman Ho and Narendra [1, 2, 3, 4], Kreindler and Sarachik [5], Lee [6], and Gilbert [7].

We shall then turn our attention to systems that are partially observable in that the output vector contains all information necessary for the unique recovery of each component of the state vector. We discuss two methods for the construction of observers, the first due to Kalman [8], and the second to Luenberger [9].
Finally, we pose the problem of combined estimation and control in which we not only have the requirement for state estimation but also the requirement to use the estimated state in such way as to generate an optimal control law. This problem has been treated by Kalman [10], Joseph and Tou [11], Gunckel and Franklin [12], and others [13, 14]. It lays the foundation for the optimal adaptive problem which we shall consider in later chapters.

11.1 Observability in linear dynamic systems

In Chapters 8, 9, and 10 we developed various concepts concerning state estimation in linear continuous and linear discrete systems. To accomplish state estimation, it is necessary that certain requirements with respect to observability be met.

For a system to be observable, it must be possible to determine the state of an unforced system from the knowledge of the output of the system over some time interval. Specifically, in an unobservable system; it is impossible to determine an initial state vector of system state variables as we are in the regulator problem. We shall first discuss the observability requirement for linear discrete systems and then proceed to a discussion of linear continuous systems.

11.1-1 Observability in time-varying discrete systems

Let us suppose that we have a system whose state is described by the unforced vector difference equation

\[ x(k + 1) = A(k)x(k) \tag{11.1-1} \]

and suppose that we observe a vector \( z(k) \) which is a linear combination of the system states plus an additive noise term

\[ z(k) = C(k)x(k) + v(k) \tag{11.1-2} \]

We desire to find the best least-squares estimate, \( \hat{x}(k) \), of \( x(k) \) by minimizing

\[ J = \frac{1}{2} \sum_{k=k_0}^k \| z(k) - C(k)\hat{x}(k) \|_2^2 \tag{11.1-3} \]

subject to the constraint of Eq. (11.1-1) with \( x(k) \) replaced by \( \hat{x}(k) \). This is a multistage decision process, and since Eq. (11.1-1) holds, we can write

\[ x(k_0 + 1) = A(k_0)x(k_0), \]
\[ x(k_0 + 2) = A(k_0 + 1)x(k_0 + 1) = A(k_0 + 1)A(k_0)x(k_0) \]

Thus it is clear that

\[ x(k_0 + k) = \varphi(k_0 + k, k_0)x(k_0) \tag{11.1-4} \]

where

\[ \varphi(k_0 + k, k_0) = A(k_0 + k - 1) \ldots A(k_0 + 1)A(k_0) = \prod_{i=k_0}^{k+k-1} A(k) \tag{11.1-5} \]

Finally, we choose \( M(k_0, k_0) = \frac{1}{2} \sum_{k=k_0}^k \| \varphi(k, k_0)x(k_0) - C(k)\hat{x}(k) \|_2^2 \tag{11.1-6} \]

Since matrix multiplication is not commutative, we realize that we must form the product in Eq. (11.1-5) in the proper order. Now we can write

\[ x(k) = \varphi(k, k_0)x(k_0) \tag{11.1-7} \]

By using Eq. (11.1-7), we can write the cost function as

\[ J = \frac{1}{2} \sum_{k=k_0}^k \| \varphi(k_0)x(k_0) - C(k)\hat{x}(k) \|_2^2 \tag{11.1-8} \]

which includes the constraint Eq. (11.1-1), since it has been used to formulate the equation.

We wish to minimize Eq. (11.1-8). To do this we will solve \( \partial J/\partial \hat{x}(k_0) = 0 \), which is the usual necessary condition for a minimum. In doing this we obtain from Eq. (11.1-8)

\[ \sum_{k=k_0}^k \varphi^T(k, k_0)C^T(k)R^{-1}(k)[\varphi(k, k_0)x(k_0) - C(k)\hat{x}(k_0)] = 0 \tag{11.1-9} \]

We note that \( \hat{x}(k_0) \) may be removed from the summation sign. By doing this and solving the resulting equation, we obtain

\[ \hat{x}(k_0) = M^{-1}(k_0, k_0) \sum_{k=k_0}^k \varphi^T(k, k_0)C^T(k)R^{-1}(k)x(k) \tag{11.1-10} \]

as the best initial condition, where we have defined

\[ M(k_0, k_0) = \sum_{k=k_0}^k \varphi^T(k, k_0)C^T(k)R^{-1}(k)C(k) \tag{11.1-11} \]

Clearly, \( M(k_0, k_0) \) must have an inverse and, therefore, must be nonsingular. Kalman’s condition for observability goes even further, in that it requires \( M(k_0, k_0) \) to be positive-definite. We recall that a positive-definite matrix \( F \) is defined as one such that \( x^TFx > 0 \) for any nonzero \( x \). Also real symmetric matrix \( F \) is positive-definite if and only if there exists a nonsingular matrix \( D \) such that \( F = D^TD \). We note that \( D \), being nonsingular, implies that \( F \) is nonsingular also, since \( \text{det}(F) = \text{det}(D))^2 \). Since \( M \) is of the form \( D^TD \), the positive-definite requirement really only requires that \( M \) be nonsingular.

For observability, we are not at all concerned with the specific nature of the positive-definite weighting matrix \( R \), and thus we set \( R = I \) in Eq. (11.1-11).

Example 11.1-1

Suppose we have two integrators in cascade as in Fig. 11.1-1a. We ask: Can we estimate \( x^2 \) by observing \( x \)? Obviously not, because we do not know the initial condition on the second integrator. In this case we would find \( M \) to be singular and thus not positive definite.

Now suppose that we add a switch to the system as shown in Fig. 11.1-1b. We begin by observing \( z^2 = [x_1, x_2] \) at some time \( t_s < t_f \). Can we estimate \( x^2 \)? We would find that \( M \) is singular for \( t > t_f \) and nonsingular thereafter, indicating that the system is observable for \( t > t_f \), and nonobservable for \( t < t_f \). This is
we can set the use of a least-squares curve fitting procedure. From Eqs. (11.1-2) and 294 could have applied to the third rank of singular for time, provided the fact that, once we know the value of what we could expect intuitively. Lastly, we add another switch, which is not necessary that \( \Delta(k_n, k_f) \) be of rank \( n \) (\( n \) is an \( n \) vector). This provides us with an alternative test for observability. If we premultiply Eq. (11.1-12) by \( \Delta(k_n, k_f) \), we have

\[
\sum_{k=k_0}^{k_f} \varphi'(k_n, k) C'(k) x(k) = \left[ \sum_{k=k_0}^{k_f} \varphi'(k_n, k) C(k) \right] x(k_0)
\]

Thus we again have

\[
\varphi'(k_n, k) C'(k) x(k) = \left[ \sum_{k=k_0}^{k_f} \varphi'(k_n, k) C(k) \right] x(k_0)
\]

where \( M(k_n, k_f) \) has been previously defined by Eq. (11.1-11). The matrix \( M(k_n, k_f) \) is sometimes called the Gram'ian matrix and is nonsingular if and only if the matrix \( \Delta(k_n, k_f) \) is of rank \( n \). Thus there certainly must be at least \( n \) columns in \( \Delta(k_n, k_f) \), which requires that the minimum sequence length, \( k_f - k_n \) is \((n^m - 1)\), where \( x \) is an \( n \) vector and \( z \) is an \( m \) vector.

For constant discrete systems where \( A \) and \( C \) are stage invariant, these results simplify somewhat since \( \varphi(k_n, k) = A^{k-n} \), \( C(k) = C \), and the observability requirement becomes that the matrix

\[
\Delta(k) = [C'] [A^n C^r] [A^n C^r] \cdots [A^n C^r]
\]

be of rank \( n \). If a constant system is not observable on a sequence of length \( k = n \), it is, of course, not observable on any sequence. This is not the case for stage-varying or nonconstant systems as indicated in Example 11.1-1. In many cases, it will be computationally more convenient to determine whether or not the \( n \times n \) matrix \( \Delta_n^r \) is of rank \( n \) rather than the \( n \times nm \) matrix \( \Delta \) of Eq. (11.1-16). This statement will apply to the many matrices of the form of Eq. (11.1-16) which we will encounter in this section and the next.

11.1-2 Observability in continuous systems

We have previously derived the observability condition for discrete static and dynamic systems. Now consider a continuous dynamic system represented by the \( n \) vector equation

\[
\dot{x}(t) = A(t)x(t)
\]

where we observe (measure) an \( m \) vector output

\[
x(t) = C(t)x(t) + v(t)
\]

where \( v(t) \) is additive measurement noise. We wish to find the best least-square estimator, \( \hat{x}(t) \) of \( x(t) \) such that the cost function

\[
J = \frac{1}{2} \int_{t_0}^{t_f} \| x(t) - C(t) \hat{x}(t) \|^2 dt
\]
We now define
\[ J = \frac{1}{2} \int_{t_0}^{t_f} \| x(t) - C(t)\phi(t, t_0)\xi(t_0) \|^2 dt \] (11.1-23)

To determine the particular \( \xi(t_0) \) that minimizes Eq. (11.1-23), we must solve
\[ \frac{\partial J}{\partial \xi(t_0)} = 0 = \int_{t_0}^{t_f} \phi^T(t, t_0)C^T(t)R^{-1}(t)(x(t) - C(t)\phi(t, t_0)\xi(t_0)) dt \] (11.1-24)
which gives
\[ \int_{t_0}^{t_f} \phi^T(t, t_0)C^T(t)R^{-1}(t)C(t)\phi(t, t_0) dt \] \[ \xi(t_0) = \int_{t_0}^{t_f} \phi^T(t, t_0)C^T(t)R^{-1}(t)x(t) dt \] (11.1-25)

We now define
\[ N(t_n, t_f) = \int_{t_n}^{t_f} \phi^T(t, t_n)C^T(t)R^{-1}(t)C(t)\phi(t, t_n) dt \] (11.1-26)
so that
\[ \xi(t_0) = N^{-1}(t_n, t_f) \int_{t_n}^{t_f} \phi^T(t, t_n)C^T(t)R^{-1}(t)x(t) dt \] (11.1-27)

Clearly, the matrix of Eq. (11.1-26) must have an inverse or, in other words, must be nonsingular. Furthermore, by computing the second derivative \( \frac{\partial^2 J}{\partial \xi^2} \), we see that we require \( N(t_n, t_f) \) to be positive-definite in order to establish sufficient conditions for a minimum of the cost function. Thus, a system becomes observable at time \( t_f \) when the matrix \( N(t_n, t_f) \) is positive-definite for \( t_n, t_f > t_0 \). Again, it can be shown that the rank of the matrix \( N(t_n, t_f) \) is nondecreasing with time. In other words, once a system becomes observable at \( t = t_0 \), it remains observable for all \( t > t_0 \). For observability, the matrix \( R \) is again set equal to the identity matrix \( I \).

We will again offer an alternate derivation of the observability requirement. The output of the system \( z(t) \) is from Eqs. (11.1-18) and (11.1-21),
\[ \dot{z}(t) = C(t)\phi(t, t_0)\xi(t_0) \] (11.1-28)
By premultiplying this equation by \( \phi^T(t, t_0)C^T(t) \) and integrating, we obtain
\[ \int_{t_n}^{t_f} \phi^T(t, t_0)C^T(t)z(t) dt = \left[ \int_{t_n}^{t_f} \phi^T(t, t_0)C^T(t)C(t)\phi(t, t_0) dt \right] \xi(t_0) \] (11.1-29)

Thus
\[ \xi(t_0) = N^{-1}(t_n, t_f) \int_{t_n}^{t_f} \phi^T(t, t_0)C^T(t)z(t) dt \] (11.1-30)
where \( N(t_n, t_f) \) is as defined before:
\[ N(t_n, t_f) = \int_{t_n}^{t_f} \phi^T(t, t_0)C^T(t)C(t)\phi(t, t_0) dt \] (11.1-31)

We can clearly solve for \( \xi(t_0) \) also by
\[ \xi(t_0) = M^{-1}(t_n, t_f) \int_{t_n}^{t_f} \phi^T(t, t_0)C^T(t)z(t) dt \] (11.1-32)
where
\[ M(t_n, t_f) = \int_{t_n}^{t_f} \phi^T(t, t_0)C^T(t)C(t)\phi(t, t_0) dt \] (11.1-33)
and we can easily show that
\[ M(t_n, t_f) = \phi^T(t_0, t_f)N(t_n, t_f)\phi(t_0, t_f) \] (11.1-34)

From Eq. (11.1-28) we see that a necessary condition for the system to be observable (on the interval \( [t_n, t_f] \)) is that the columns of \( C(t)\phi(t, t_0) \) be linearly independent. Mathematically, we may write this condition of linear independence in terms of an \( m \times 1 \) vector \( \eta \) as [15, 16]
\[ \eta^T C(t)\phi(t, t_0) \neq 0 \quad \forall t \in [t_n, t_f], \quad \eta \neq 0 \] (11.1-35)
This condition may be developed into a test for observability as follows. If we assume that the conditions of Eq. (11.1-35) are not fulfilled, and differentiate Eq. (11.1-35) repeatedly, noting that \( \frac{\partial \phi(t, t_0)}{\partial t} = A(t)\phi(t, t_0) \), we obtain the set of equations
\[ \eta^T \Gamma_j(t)\phi(t, t_0) = 0, \quad j = 1, 2, \ldots, n \] (11.1-36)
where
\[ \Gamma_j = C^T(t) \]
\[ \Gamma_k = \frac{\partial \Gamma_{k-1}}{\partial t} + A^T(t)\Gamma_{k-1} \] (11.1-37)
Now if we define
\[ \Gamma = [\Gamma_1, \Gamma_2, \ldots, \Gamma_n] \] (11.1-38)
we have that for \( m \) vectors which we call \( \alpha \) we have
\[ \alpha^T \Gamma(t) \varphi(t, t) = 0^m \] (11.1-39)
which, since \( \varphi \) is always nonsingular, implies that \( \Gamma \) is singular. But Eq. (11.1-35) does not express an equality, so none of these relations, Eq. (11.1-36), could hold, and \( \Gamma \) cannot be singular if the system is observable. Thus, if the \( \Gamma \) matrix of Eq. (11.1-38) is of rank \( n \), where \( \Gamma_2 \) is defined in Eq. (11.1-37), the system is observable.

The matrices \( M(t_0, t_2) \) and \( N(t_0, t_2) \) are known as Gramian matrices and must be positive-definite for an observable system. This is an alternate and equivalent criterion to requiring the \( \Gamma \) matrix to be of rank \( n \). For a constant system, it is considerably simpler to determine the rank of the \( \Gamma \) matrix than to evaluate either of the Gramian matrices. Thus for a constant system, the easiest criterion for observability is to use the requirement that the \( n \times n \) matrix
\[ \Gamma = [C^T \mid A^T C^T \mid A^T C^T \mid \cdots \mid A^T C^T] \] (11.1-40)
be of rank \( n \). This may be accomplished if we determine whether the \( n \times n \) matrix \( \Gamma \Gamma^T \) is of rank \( n \).

We may now distinguish between several types of observability. A system is said to be observable on the interval \( [t_0, t_2] \) if, for a specified \( t_0 \) and specified \( t_2 \), every state \( x(t_0) \) may be determined from knowledge of \( z(t) \) \( t \in [t_0, t_2] \). In other words, the \( M \) matrix is positive-definite or the rank test is satisfied for the fixed \( t_0 \) and fixed \( t_2 \). If this is true for all \( t_0 \) and some \( t_2 > t_0 \), we say that the system is completely observable. If this is true for every \( t_0 \) and every \( t_2 > t_0 \) the system is said to be totally observable. The only modification to this statement needed to treat discrete systems is that there are a finite number of states, as discussed in Section 11.1-1, before a discrete system will become observable. Finally, we remark that application of the state estimation techniques of the previous two chapters to unobservable systems often leads to impossible computational problems in determining the solution to the error variance equation. A remedy is to attempt to estimate only those components of the state vector which are observable in the output vector.

### 11.2 Controllability in linear systems

In Chapters 9 and 10, we saw that the linear state estimation and the regulator problem were duals of one another. Thus it is reasonable to expect a dual of the observability criterion, and we shall call it the controllability criterion. We will say that a system is state controllable if any initial state vector \( x(t_0) \) can be transferred to any final state vector \( x(t_f) \), where \( t_0 \) and \( t_f \) are fixed by means of some control \( u(t) \). More precise definitions of controllability, as well as a discussion of the implications of duality, will be given at the end of this section. We shall first consider state controllability as output controllability for continuous systems. The close similarity of the results will then be noted. As suits the dual to observability, we shall initiate our approach by considering the transfer of the system from the initial state to a final state which, since linear systems are being considered, can be considered to be the origin without loss of generality.

Suppose we wish to determine whether the system described by
\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{11.2-1} \]
\[ x(t_0) = x_0 \tag{11.2-2} \]
is controllable. In other words, we wish to find whether there is a control \( u(t) \), such that \( x(t_f) = x_f \) and \( x(t_f) = 0 \). We will find the control which accomplishes this (if it exists) and which minimizes the cost function
\[ J = \frac{1}{2} \int_{t_0}^{t_f} \| u(t) \|_2^2 dt \tag{11.2-3} \]

We will use this cost function to "get a handle" on the problem, i.e., to determine if there is a \( u(t) \) such that we can bring the system from \( x(t_0) = x_0 \) to \( x(t_f) = 0 \). Another "sensible" cost function would work equally well. To do this, we shall use the maximum principle. Thus, we form the Hamiltonian
\[ H[x(t), u(t), \lambda(t), t] = \frac{1}{2} \| u(t) \|_2^2 + \lambda^T(t)[A(t)x(t) + B(t)u(t)] \tag{11.2-4} \]
and obtain in the usual way
\[ \frac{\partial H}{\partial \lambda} = \dot{x} = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \tag{11.2-5} \]
\[ \frac{\partial H}{\partial x} = -\dot{\lambda} = A^T \lambda(t), \quad x(t_f) = 0 \tag{11.2-6} \]

To obtain the minimum \( H \), we set
\[ \frac{\partial H}{\partial u} = 0 \tag{11.2-7} \]

which gives
\[ u(t) = -R^{-1}(t)B^T(t)\lambda(t) \tag{11.2-8} \]

By combining these last four equations, we obtain
\[ \dot{x} = A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda(t), \quad x(t_0) = x_0 \tag{11.2-9} \]
\[ \dot{\lambda} = -A^T(t)\lambda(t), \quad x(t_f) = 0 \tag{11.2-10} \]
In a fashion similar to that which we have used many times before, we obtain the solution to these two equations as

\[ x(t) = \varphi(t, t_0)x(t_0) - \int_{t_0}^{t} \varphi(t, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\lambda(\tau) d\tau \]  

(11.2-11)

\[ \lambda(t) = \varphi^{T}(t, t_0)\lambda(t_0) \]  

(11.2-12)

By combining Eqs. (11.2-11) and (11.2-12), we obtain

\[ x(t) = \varphi(t, t_0)x(t_0) - \int_{t_0}^{t} \varphi(t, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\lambda(\tau) d\tau \]  

(11.2-13)

which must be zero. An alternate approach is to write

\[ x(t) = \int_{t_0}^{t} \varphi(t, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\lambda(\tau) d\tau \]  

(11.2-14)

which, since \( x(t_0) = 0 \), becomes just

\[ x(t) = -\int_{t}^{t_0} \varphi(t, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\lambda(\tau) d\tau \]  

(11.2-15)

But, since

\[ \lambda(t) = \varphi^{T}(t, t_0)\lambda(t_0) \]  

(11.2-16)

Eq. (11.2-15) can be written, if we choose \( t = t_0 \), as

\[ x(t_0) = \int_{t}^{t_0} \varphi(t, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\lambda(\tau) d\tau \]  

(11.2-17)

Now we can solve either Eq. (11.2-13) for \( \lambda(t_0) \) or Eq. (11.2-17) for \( \lambda(t_0) \). Suppose we choose the latter. Then

\[ \lambda(t_0) = -W^{-1}(t_0, t_0)x(t_0) \]  

(11.2-18)

where

\[ W(t_0, t_0) = \int_{t_0}^{t} \varphi(t, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\lambda(\tau) d\tau \]  

(11.2-19)

If a system is state controllable, \( W(t_0, t_0) \) must have an inverse and also be positive-definite as the second variation would show. Again \( R \) may be set equal to the identity matrix. In Section 9.2, we had a relation very similar to Eq. (11.2-19), which we converted to a differential equation. We found that it was very much easier to solve the differential equation than to evaluate the integral. Let us now try the same approach here. Differentiation of Eq. (11.2-19) gives

\[ \frac{\partial W(t_0, t)}{\partial t_0} = -\varphi(t_0, t_0)B(t_0)R^{-1}(t_0)B^{T}(t_0)\lambda(t_0, t) \]  

\[ + \int_{t_0}^{t} \varphi(t_0, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\varphi^{T}(\tau, t_0) d\tau \]  

\[ + \int_{t_0}^{t} \varphi(t_0, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau) \frac{\partial \varphi^{T}(\tau, t_0)}{\partial t_0} d\tau \]  

(11.2-20)

which becomes, since \( \varphi(t_0, t_0) \partial t = A(t_0)\varphi(t_0, t_0) \) and \( \varphi(t, t) = I \),

\[ \frac{\partial W(t_0, t)}{\partial t_0} = -B(t_0)R^{-1}(t_0)B^{T}(t_0) \]  

\[ + A(t_0) \int_{t_0}^{t} \varphi(t_0, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\varphi^{T}(\tau, t_0) d\tau \]  

\[ + \int_{t_0}^{t} \varphi(t_0, \tau)B(\tau)R^{-1}(\tau)B^{T}(\tau)\varphi^{T}(\tau, t_0)A^{T}(\tau) d\tau \]  

(11.2-21)

But, by Eq. (11.2-19), the two integrals are just \( W(t_0, t_0) \). Therefore,

\[ \frac{\partial W(t_0, t_0)}{\partial t_0} = -B(t_0)R^{-1}(t_0)B^{T}(t_0) \]  

\[ + W(t_0, t_0)A^{T}(t_0), \quad W(t_0, t_0) = 0 \]  

We have, therefore, succeeded in obtaining a differential equation for \( W(t_0, t_0) \) which should be easier to solve than the defining relation for \( W(t_0, t_0) \) Eq. (11.2-19).

It is interesting now to evaluate the cost function of Eq. (11.2-3) which by Eq. (11.2-8), becomes

\[ J = \frac{1}{2} \int_{t_0}^{t} \lambda^{T}(t)B(t)R^{-1}(t)B^{T}(t)\lambda(t) dt \]  

(11.2-22)

But \( R(t) \) is symmetric, so that

\[ J = \frac{1}{2} \int_{t_0}^{t} \lambda^{T}(t)B(t)R^{-1}(t)B^{T}(t)\lambda(t) dt \]  

(11.2-23)

From Eqs. (11.2-16) and (11.2-18), we see that

\[ \lambda(t) = \varphi^{T}(t_0, t)\lambda(t_0) \]  

(11.2-24)

From the defining relation for \( W(t_0, t_0) \), we know that it is symmetric; hence Eq. (11.2-24) becomes

\[ J = \frac{1}{2} \int_{t_0}^{t} \lambda^{T}(t)W^{-1}(t_0, t)B(t)R^{-1}(t)B^{T}(t)\lambda(t) dt \]  

(11.2-25)

By excluding those terms from the integral which do not involve \( t \), we get

\[ J = \frac{1}{2} \int_{t_0}^{t} \lambda^{T}(t)W^{-1}(t_0, t)\lambda(t) dt \]  

(11.2-26)

Or, since the integral in the brackets is just the definition of \( W(t_0, t_0) \), we have finally

\[ J = \frac{1}{2} \lambda^{T}(t)W^{-1}(t_0, t)\lambda(t) = \frac{1}{2} \| x(t_0) \|_{W^{-1}(t_0, t)}^{2} \]  

(11.2-27)

Equation (11.2-28) allows an interesting interpretation of controllability. Suppose that we are given some definite value for the cost \( J \). Then, if we can determine \( W^{-1}(t_0, t) \), we can find all initial conditions such that \( J \)}
(11.2-28) is satisfied. We can thus plot a surface in n-space representing those initial conditions from which we can take the system to the origin with a cost of J. This problem is known as the reachable zone problem, which is considered in Problem 4 of this chapter.

We can offer an alternative approach to this problem. We shall do this for the output controllability problem which reduces to the state controllability problem when C(t) = I. The solution to Eqs. (11.2-1) and (11.2-2) is the m vector output due to the r vector control

$$z(t) - C(t)p(t, t_0)x(t_0) = C(t) \int_{t_0}^{t} \varphi(t, \tau)B(\tau)u(\tau) d\tau$$  \hspace{1cm} (11.2-29)

At time t, the left-hand side of this equation is simply equal to some specified value z_d(t) such that we may write

$$z_d(t) = z(t) - C(t)p(t, t_0)x(t_0) = \int_{t_0}^{t} C(t)\varphi(t, \tau)B(\tau)u(\tau) d\tau$$  \hspace{1cm} (11.2-30)

A sufficient condition for output controllability on [t_0, t] is that the columns of C(t)\varphi(t, \tau)B(\tau) be linearly independent, which means that, for arbitrary m vector \eta, we have the r vector equation [15, 16]

$$\eta^T C(t)\varphi(t, \tau)B(\tau) \neq 0^r, \quad t_0 \leq \tau \leq t$$  \hspace{1cm} (11.2-31)

We may develop another output controllability condition from this condition. This proof will proceed by the method of contradiction. Suppose that there exists at least one nonzero vector \eta, such that Eq. (11.2-31) is, in fact, true. Repeated differentiation of Eq. (11.2-31) with respect to \tau yields

$$\eta^T C(t)\varphi(t, \tau)\Gamma(\tau) = 0^r, \quad j = 1, 2, \ldots, n$$  \hspace{1cm} (11.2-32)

where, since \varphi(t, \tau)/\partial\tau = -\varphi(t, \tau)A(\tau),

$$\Gamma(\tau) = B(\tau)$$

$$\Gamma(\tau) = \frac{\partial \Gamma_{2-1}(\tau)}{\partial \tau} - A(\tau)\Gamma_{2-1}(\tau)$$  \hspace{1cm} (11.2-33)

Then, if we define the n by nm matrix \Gamma

$$\Gamma = [\Gamma_1, \Gamma_2, \ldots, \Gamma_n]$$  \hspace{1cm} (11.2-34)

the condition of Eq. (11.2-32) becomes, for the m vectors \eta',

$$\eta'^T C(t)\varphi(t, \tau)\Gamma = 0^r$$  \hspace{1cm} (11.2-35)

which would tell us that \Gamma could not be of rank n since \eta is nonsingular (excluding for the moment the possibility of C being singular). But Eq. (11.2-35) cannot be zero by Eq. (11.2-31), and so \Gamma must then be of rank \eta, and Eq. (11.2-35) will not, in fact, be zero. Although this requirement holds for time-varying systems, it is particularly easy to apply in the case of constant systems, for then, as is easily verified, for \Gamma' = [\Gamma_1, -\Gamma_2, \ldots, (-1)^{n-1}\Gamma_n],

$$\Gamma' = [B | AB | A^2B | \ldots | A^{n-1}B]$$  \hspace{1cm} (11.2-36)

and this must be of rank n. This is only the requirement for state controllability since, if a constant system is controllable at all, it is control at \tau = t_0 (impulse control required). Therefore, from Eq. (11.2-35), the put controllability requirement is that

$$[CB, CAB, CA^2B, \ldots, CA^{n-1}B]$$  \hspace{1cm} (11.2-37)

be of rank m. For the general time-varying case, the C(t)\Gamma term of (11.2-35) must be of rank m since we know that \eta must be nonsingular.

If, in Eq. (11.2-30), we let

$$u(t) = B(t)\varphi(t, t_0)x(t_0), \quad \lambda(t) = -V^{-1}(t_0, t)z_d(t_0)$$  \hspace{1cm} (11.2-41)

where

$$V(t_0, t_1) = \int_{t_0}^{t_1} C(t)\varphi(t, \tau)B(\tau)B^T(\tau)\varphi^T(t, \tau) d\tau$$  \hspace{1cm} (11.2-42)

and must be positive-definite for a controllable system. For state controllability, we may treat C = I; then we can easily show that

$$V(t_0, t_1) = \varphi(t_1, t_0)W(t_0, t_1)\varphi^T(t_1, t_0)$$  \hspace{1cm} (11.2-43)

where W(t_0, t_1) is defined by Eq. (11.2-19).

It is quite easy for us to show that all of these results carry over to the discrete system described by

$$x(k + 1) = A(k)x(k) + B(k)u(k)$$

$$x(k) = C(k)x(k)$$

except that discrete transition matrices and summations are used rather than continuous transition matrices and integrations. The time interval \tau is then replaced by the sequence k_0, k_0 + 1, \ldots, k. Thus, for instance, discrete equivalent of Eq. (11.2-19) is

$$W(k_0, k_f) = \sum_{k=k_0}^{k_f} \varphi(k, k_0)B(k)R(k)B^T(k)\varphi^T(k, k)$$  \hspace{1cm} (11.2-46)

Analogous to the discrete observability requirement, a controllable discrete system can be transferred to the origin in at most n stages, where x is vector.

Just as in the case of observability, there are several different types of controllability. We will give these definitions for the case of state controllability. Output controllability definitions follow merely by replacing x(t_0) by u(t) in the definitions.

We will say that a system is state controllable for a given t_0 and t_f if initial state x(t_0) can be transferred to any final state x(t_f) using any control u(t) over the interval [t_0, t_f]. A system will be said to be completely state
Controllable if, for any \( t_s \), each initial state \( x(t_s) \) can be transferred to any final state and given final time \( x(t_f) \) where, of course, \( t_f \geq t_s \). To obtain total state controllability, the system must be completely state controllable for every \( t_s \) and every \( t_f \).

Example 11.2-1

Let us consider the linear system described by

\[
\begin{align*}
\dot{x}_1 &= x_2(t) + u(t), \\
\dot{x}_2 &= -x_1(t) - 2x_2(t) - u(t), \\
z_1(t) &= x_1(t) \\
z_2(t) &= x_1(t) + x_2(t)
\end{align*}
\]

The system dynamics can also be written as

\[
\dot{x} = Ax(t) + bu(t), \quad z(t) = Cx(t)
\]

where

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

We wish to determine the observability and controllability of the system. From the preceding section we know that the system is observable if the \( n \times nm \) matrix

\[
[C^t | A^TC^t | \cdots | A^{n-1}C^t] = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}
\]

is of rank 2. This is the case, and so the system is observable. To discern state controllability, we must examine the matrix

\[
[B | AB | A^2B | \cdots | A^{n-1}B] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

to see if it is of rank 2. Clearly it is not, and so this system is not state controllable. Neither is the system output controllable, because the matrix

\[
[CB | CAB | CA^2B | \cdots | CA^{n-1}B] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}
\]

is not of rank 2.

Let us now examine the reasons for this uncontrollability. Figure 11.2-1 illustrates a possible block diagram for this system. Appropriate transfer functions for the system are

\[
\frac{x_1(s)}{u(s)} = \frac{1}{s + 1}, \quad \frac{x_2(s)}{u(s)} = \frac{-1}{s + 1}
\]

and we observe that the physical reason the system is not state controllable is that the state vector \( x(t) \) can be controlled only along or parallel to a straight line \( x_1(t) + x_2(t) = 0 \). This is certainly not in two dimensions; therefore the system is not state controllable. Appropriate transfer functions for the output state are

\[
\frac{z_1(s)}{u(s)} = \frac{x_1(s)}{u(s)} = \frac{1}{s + 1}, \quad \frac{z_2(s)}{u(s)} = \frac{x_1(s) + x_2(s)}{u(s)} = 0
\]

Since the output \( z(t) \) cannot be controlled by the input, the entire system is not output controllable. If the output were just \( z_1(t) \), a scalar, then the system is not state controllable but is output controllable. This means that we could determine an input which could drive \( z_1(t) \) to any given value but could not drive \( x_1(t) \) and \( x_2(t) \) to any value which lies off the line \( x_1(t) + x_2(t) = 0 \). We note that we were given a second order system but found first order transfer function from control input to state and output state variables. This implies that that the given system is "reducible" in order. Choate and Sage [16] have shown that systems which are not totally controllable must be reducible.

Earlier we remarked that the dual of an unobservable system is an uncontrollable system. This can easily be seen if we observe the observability criteria where the adjoint system \( A^* = -A^T, \ B^* = C^*, \ C^* = B^T \) is used and if we note that the observability criteria becomes the controllability criteria. Thus we may say that a system is controllable if the adjoint system is observable. Since the dual system is defined by \( A^*(t^*) = A^T(t), \ B^*(t^*) = C^T(t), \ C^*(t^*) = B^T(t), \ t^* = -t \), we see that the similar statement for dual systems, a system is uncontrollable (unobservable) if its dual is unobservable (controllable), applies.

For successful control, it is normally necessary that systems be both controllable and observable. For example, if a subsystem which is unobservable is part of a closed-loop system, instabilities in the unobservable part of the system cannot be detected or stabilized by the closed loop. If a system is not state controllable, it is not possible to control a portion of the system, and thus persistent transients may exist. If the system is not output controllable, then it appears that all is lost unless it is possible to change input and/or output state variables.

Even though a system may be observable, not all components of the state variable, \( x(t) \), may be recoverable immediately from the observation \( z(t) \). We recall that \( x(t) \) may well be a scalar, \( x(t) \) may well be a 100 vector, and the system may certainly be observable. In the next section we shall discuss methods of state-variable recovery from observable output vectors.
Lagrange Multipliers.

Where \( p(t) \) are the \((n)\)

\[ p(t) \left[ a(x', y') \cdot x' \right] = \frac{\alpha}{\beta} \]

\[ j(0) = \frac{\beta}{\gamma} + \frac{\gamma}{\delta} + \frac{\delta}{\epsilon} \]

We take the Augmented J as

\[ 2 \partial \left[ \frac{\partial \psi}{\partial x'} + \frac{x}{\partial x'} + \frac{\partial \psi}{\partial y'} \right] \]

Thus

\[ \frac{\partial \psi}{\partial x'} = (t, x') \]

\[ \frac{\partial \psi}{\partial y'} = (t, y') \]

\[ \frac{\partial \psi}{\partial z'} = (t, z') \]

and we can drop it.

Since \( h(x, x', t) \) is constant

\[ \int_{t}^{0} x' \cdot h \]

\[ \int_{t}^{0} y' \cdot h \]

\[ \int_{t}^{0} z' \cdot h \]

Now

\[ x = a \times x \text{ vector} \]

subject to \( x(t) = a \cdot (x(t), y(t), z(t)) \)

\[ j(t) = h(x(t), y(t), z(t)) \]

General problem is to minimize

I Necessary Conditions for Optimal Control

Control Problems

The Variational Approach to Optimal
\[ 0 = 2p \left\{ \int_{0}^{1} x^2 \left[ \frac{e^x}{x^2} + \frac{e^{\frac{x^3}{3}}} {x^2} \right] \ dx \right\} \]

Thus

\[ 0 = \frac{3}{4} x^2 - \frac{3}{4} x^2 + \frac{3}{4} x^2 - \frac{3}{4} x^2 \]

Applying Chain Rule to last term:

\[ \frac{x}{x^2} + \frac{3}{x^3} + \frac{x}{x^2} \]

or

\[ \frac{1}{x^2} \int \left[ \frac{e^x}{x^2} + \frac{e^{\frac{x^3}{3}}} {x^2} \right] \ dx \]

AR BE DEPENDENT ON \( y \), INSIDE INTEGRAL ARE

Terms in this expression which

specification of \( t \) and \( x(t) \). The

Note that we have allowed non-

\[ \frac{d}{dt} \left\{ 2p \left[ \int_{0}^{1} x^2 \left[ \frac{e^x}{x^2} + \frac{e^{\frac{x^3}{3}}} {x^2} \right] \ dx \right\} \right\} \]

Trick gives the variation

the standard parts integration

of \( u \) and \( p \), and performing

recognizing that \( g \) is independent

\[ u(0) = \left. \int_{0}^{t} \right] \]

Minimizing

Our problem now becomes one of

\[ \frac{d}{dt} \left[ \frac{e^x}{x^2} + \frac{e^{\frac{x^3}{3}}} {x^2} \right] \]

Define auxiliary
More on this later.

\[ 0 = \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial x} \left( \frac{1}{2} v^2 + \frac{1}{4} p \right) \right] + \frac{1}{2} v \frac{\partial p}{\partial t} \]

The boundary condition:

We have yet to deal with important.

These three equations are:

\[ p = 0 = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \right) \right) + \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \right) \]

Likewise, \( \partial \) zero. Thus the coefficient \( \partial \) 0 must this is the constant equation.

\[ p(t) = \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \right) \]

Coefficient of \( \partial \) is zero.

Choose \( p \) such that the coefficient of \( \partial \) is zero.

\[ \partial \]

First off refer to this equation.

Necessary conditions with.

We are ready to make our
We may simplify the above equations by introducing the Hamiltonian

\[ \mathcal{H}[x, u, p, t] = g[x, u, t] + p^T(t) q(x, u, t) \]

Our equations become, \( \forall t \in [t_0, t_f] \)

\[ \begin{align*}
  \dot{x} &= \frac{\delta \mathcal{H}}{\delta p} \\
  \dot{p} &= -\frac{\delta \mathcal{H}}{\delta x} \\
  0 &= \frac{\delta \mathcal{H}}{\delta u} \\
  [\frac{\delta \mathcal{H}}{\delta x} - p]_{t_f} + [\mathcal{H}]_{t_f} + \left[ \frac{\delta \mathcal{H}}{\delta t} \right]_{t_f} &\Delta t_f = 0
\end{align*} \]
Now, let's generalize.

We must satisfy this and

We substitute this into

\[ \frac{\partial f}{\partial x} = \frac{6}{x} \]

Thus, the normal to \( \Gamma \) are
to first order, normal

The permissible values of \( x(t) \)

This is a circle.

Example: \( m(x) = (x^2 - 1)^2 + (x^2 - 1)^2 \)

Final state on surface \( m(x_c) = 0 \)

0, \( x(t) = 0 \)

\( \frac{dx}{dt} \) fixed

A Boundary Conditions
Performance measures

\[ T = \text{has rank n} \]

\[ G = \begin{bmatrix} C_0 \mid A \mid C_1 \mid A^2 \mid C_2 \mid \ldots \mid A^n \mid C_n \end{bmatrix} \]

For linear time-invariant systems

\[ x = Ax + Bu \]

\[ \begin{bmatrix} A_0 \mid A_1 \mid A_2 \mid \ldots \mid A_{n-1} \mid B \end{bmatrix} \]

For linear time-invariant systems
\[ \begin{align*}
\frac{2}{t} \cdot \frac{x}{t} &= \frac{d}{d} = \frac{x}{t} \\
q + \frac{1}{t} &= \frac{x}{t} & \text{Thus:} \\
q + \frac{t}{x} &= \frac{x}{t} & \text{NOW} \\
O &= \frac{x}{t} = \frac{1}{4} \text{ instead of } \frac{1}{5} \\
&
\end{align*}\]

\[ x + 1 = \left( x + 5 \right) \left( x + \right) \]

\[ 0 = \frac{x}{x} + 1 \]

\[ \frac{x}{t} \cdot \frac{1}{x} = \left( \frac{2}{t} \right) \Theta \]

\[ \frac{t}{x} + 1 = \frac{t}{x} + \left( \frac{2}{t} \right) \Theta \]

\[ O = \frac{x}{t} = \frac{1}{4} \]

\[ \frac{x}{t} + t = \frac{1}{4} \]

\[ \text{Example:} \]

\[ \frac{t}{x} \cdot \left( \frac{2}{t} \right) = \left( \frac{2}{t} \right) \Theta \]

\[ \text{FOR TERMINAL STATE SPECIFIED:} \]

\[ O = \frac{x}{t} + t \]

\[ \text{TRANSVERSALITY CONDITION:} \]

\[ O = \frac{t}{x} - \frac{t}{x} \]

\[ \text{EULER'S EQUATION:} \]

\[ \frac{2}{t} \cdot \left( \frac{2}{t} \right) + \left( \frac{2}{t} \right) + \left( \frac{2}{t} \right) \]

\[ \text{THEN} \]

\[ \frac{t}{x} \cdot \left( \frac{2}{t} \right) = \frac{t}{x} \]

\[ \text{VARATIONAL CALCULUS:} \]

\[ \frac{t}{x} \cdot \left( \frac{2}{t} \right) = \frac{t}{x} \]

\[ \text{}\]
Derivatives are continuous.

We can have no corners since

$$\frac{dx}{dt} = \frac{x}{t}$$

So, we require

$$x(t) = 0$$

Thus, the solution is

$$x = 0$$

Example

$$J = \int \left( \frac{d}{dt} \left( \int (1-x^2) \frac{dx}{\phi} \right) \right)^2 dt$$

Lagrange's 1st Variation Conditions

Must also satisfy Euler

$$\frac{d}{dx} \left( \frac{\phi}{\phi x} \right) = \phi \frac{d}{dx}$$
\[ C = \{ x \in \mathbb{R} \mid \phi(x) = 0, \theta(x) = 0 \} \]

\[ \phi(x) = x^2 + y^2 \leq 9 \]

\[ \theta(x) = x^2 - xy + y^2 = 0 \]

\[ \Rightarrow \begin{cases} x^2 + y^2 = 9 \\ x^2 - xy + y^2 = 0 \end{cases} \]

Let \( \mathbf{W} = \begin{pmatrix} w \\ v \end{pmatrix} \)

\[ \mathbf{W} = \begin{pmatrix} 1 \end{pmatrix} \]

\[ \mathbf{W} = \begin{pmatrix} w \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Example: \( J = \int f(x, y, t) \)

\[ \text{Use same equations} \]

\[ \text{Differential constraints} \]

\[ \text{will solve for} \quad \phi \]

\[ \begin{pmatrix} \partial \phi \\ \partial \theta \end{pmatrix} \]

\[ \text{Euler's equation is} \quad \phi \]

\[ \begin{pmatrix} P \phi \end{pmatrix} \]

\[ \text{Use Lagrange multipliers and write} \]

\[ \mathbf{W} = \mathbf{u} + \mathbf{v} \]

\[ \text{Select to} \quad \mathbf{W} \]

\[ \text{Minimize} \quad J(\mathbf{W}) = \int f(t, \mathbf{W}, \dot{\mathbf{W}}) \]

\[ \text{Point extrema} (\text{point constraints}) \]
\[ \lambda = \text{CONST} \]

\[ \nabla \cdot \mathbf{m} = \rho \frac{\partial \phi}{\partial t} \]

\[ \nabla \cdot \mathbf{m} = \rho \frac{\partial \phi}{\partial t} \]

\[ \mathbf{m} = \rho \frac{\partial \phi}{\partial t} \]

\[ \mathbf{m} = \rho \frac{\partial \phi}{\partial t} \]

**Example:**

**Note:**

\[ \phi = \phi_0 \]

\[ \mathbf{T} = \mathbf{T}_0 \]

\[ \mathbf{m} = \mathbf{m}_0 \]

\[ \mathbf{u} = \mathbf{u}_0 \]

**Subject to:**

\[ \mathbf{u} = \mathbf{u}_0 \]

\[ \mathbf{m} = \mathbf{m}_0 \]

\[ \mathbf{T} = \mathbf{T}_0 \]

\[ \phi = \phi_0 \]

\[ \mathbf{m} = \mathbf{m}_0 \]

\[ \mathbf{T} = \mathbf{T}_0 \]

**Inequality Constraints:**

\[ \int_{\text{MAX}}^{\text{MIN}} \left( \mathbf{w}_m - \mathbf{w} \right) \mathbf{d} \mathbf{T} = 0 \]

\[ \int_{\text{MIN}}^{\text{MAX}} \left( \mathbf{w}_m - \mathbf{w} \right) \mathbf{d} \mathbf{T} = 0 \]

\[ \int_{\text{MIN}}^{\text{MAX}} \left( \mathbf{w}_m - \mathbf{w} \right) \mathbf{d} \mathbf{T} = 0 \]

\[ \int_{\text{MIN}}^{\text{MAX}} \left( \mathbf{w}_m - \mathbf{w} \right) \mathbf{d} \mathbf{T} = 0 \]

\[ \int_{\text{MIN}}^{\text{MAX}} \left( \mathbf{w}_m - \mathbf{w} \right) \mathbf{d} \mathbf{T} = 0 \]

\[ \int_{\text{MIN}}^{\text{MAX}} \left( \mathbf{w}_m - \mathbf{w} \right) \mathbf{d} \mathbf{T} = 0 \]
must integrate backward.

\[ \lambda(t) = 5 \quad \text{Thus, we} \]
\[ P = \sigma - \lambda \lambda \tau - \tau \]

This gives us the Riccati Eq.

The coefficient must be zero.

\[ \begin{bmatrix} P + \sigma \lambda - \lambda \sigma - \tau P \tau \end{bmatrix} = 0 \]

Combining these, we have

And, \( \lambda = \sigma \lambda + P \lambda = -\sigma \lambda + \lambda P \times 1 \)

Then, \( \lambda = \sigma - \lambda \tau \times 1 \)

Assume that \( \lambda(t) = P(t) x(t) \)

\[ x = u \tau + \sigma \lambda \]

Also, we have the terminal condition:

\[ x = x(t) = \begin{bmatrix} \sigma x(t) \\ \sigma \end{bmatrix} \]

Apply Minimum Principle:

\[ H(x, u) = \begin{bmatrix} \frac{1}{2} x \sigma x + \tau u + \lambda \tau \sigma \end{bmatrix} \]

\[ 1 = H(x, u) = \begin{bmatrix} \frac{1}{2} x \sigma x + \tau u \end{bmatrix} \]

The Linear Regulator.
\[ p + \frac{1}{2} p = P \]  
\[ p + \frac{1}{2} p = P \]  
\[ p = P_{1} A P_{1} + P_{1} b P_{1} + \frac{1}{2} P \]  
\[ x_{1} = \frac{1}{2} p \]  
\[ x_{1} = \frac{1}{2} p \]  
\[ E_{1} \]  
\[ E_{1} \]  
\[ \text{Example} \]  
\[ \text{With } p = S \]  
\[ p = A P_{1} + P_{1} A = E P_{1} + P_{1} g p_{1} \]  
Using this on our original equation:  
\[ p p_{1} + p p_{1} = c \rightarrow \]  
\[ p p_{1} + p p_{1} = c \rightarrow \]  
\[ \text{objective function:} \]  
\[ \text{objective function:} \]  
\[ p_{1} = I \]  
\[ p_{1} = I \]  
\[ \text{K(t): is the Kalman Gain, now} \]  
\[ v(t) = K(t) x(t) \]  
\[ v(t) = K(t) x(t) \]  
\[ \text{Assume a Feedback Relation:} \]  
\[ \text{Assume a Feedback Relation:} \]
Perform variation in $t_0$

Intergrate by parts:

Define the Hamiltonian:

Subject to $x = \phi(x', y', t)$, $x = \Theta(x', y', t)$

A fixed beginning and terminal times

The Bolza Problem
Our system of equations are thus:

\[ \begin{align*}
    0 &= c_1 x^2 + c_2 x + c_3 \\
    0 &= c_1 x^2 + c_2 x + c_3 \\
    0 &= c_1 x^2 + c_2 x + c_3 \\
    \end{align*} \]

Example:

Terminal manifold: \( x_1(c) + x_2(c) = 0 \)

\( x_1(c) = 0 \)
\( x_2(c) = 0 \)
Thus, in addition to our previous constraints we have:

\[ y(t) = \frac{5x}{6} + \frac{5}{6} x^{\frac{5}{2}} + \frac{1}{3} x^{3/2} + \frac{1}{5} x^{5/2} + \frac{1}{7} x^{7/2} + \frac{1}{9} x^{9/2} \]

Proceeding as above give:

\[ t = \sqrt{\frac{H}{g}} \]

\[ \mathbf{v} = \mathbf{u} + \mathbf{a} \cdot t \]

\[ \mathbf{r} = \mathbf{r}_0 + \mathbf{v} \cdot t \]

At the moment 0:

\[ \mathbf{v} = 0, \quad \mathbf{r} = \mathbf{r}_0 \]

Same problem with initial and terminal constraints.
Given\
\[X = 6 + 4\sqrt{3} + 4\, x - (x - 6)/\sqrt{3}\]\

Rearranging, and recognizing that
\[5 = \frac{3}{\sqrt{3}}\, x + \frac{\sqrt{3}}{3}\, (x + 6) + \frac{\sqrt{3}}{3}\, (x + 6)^2\]

Perform variation:
\[2p\left[ x + H \right] + \frac{\sqrt{3}}{3}\, (x + 6)^2\]

Let \(\Theta = 0\)

Perform an integration by parts:
\[2p\left[ x + H \right] + \frac{\sqrt{3}}{3}\, (x + 6)^2\]

Let \(\Theta = 0\)

At the beginning, if is free, x(t) is free.
\[ 0 = \frac{1}{2} \frac{d^2 \theta}{d \alpha^2} + \frac{1}{2} \frac{d^2 \phi}{d \alpha^2} + H \theta \]

\[ \dot{\theta} = N \]

\[ \dot{\phi} = 8 \times \dot{\alpha} \]

**Transverse Flatlet Conditions**:

\[ \begin{align*}
\frac{x_1}{\theta^2} + \frac{x_2}{\theta^2} &= 0 = \frac{x_3}{\theta^2} \\
\frac{x_1}{\phi^2} + \frac{x_2}{\phi^2} &= 0 = \frac{x_3}{\phi^2} \\
\dot{t} &= \dot{x} = \frac{x_1}{\theta} \\
\dot{t} + \dot{y} + \phi &= H = 0 \\
\end{align*} \]

Setting \( S_1 = 0 \) gives...
Transversality Conditions Become

\[ 0 = \frac{\partial L}{\partial \theta} + \gamma L \]

Let \( \theta = \gamma L \)

\[ 0 = \frac{\partial L}{\partial \theta} + \gamma L \]

Euler-Lagrange Becomes

\[ H - \frac{\partial}{\partial x^T} \int_{t_0}^{t} \left[ \frac{\partial}{\partial \theta (\theta)} \frac{\partial}{\partial x^T} \right] \Rightarrow \text{Lagrangean} \]

\[ 0 = \frac{\partial}{\partial x^T} \int_{t_0}^{t} \left[ \frac{\partial}{\partial \theta (\theta)} \frac{\partial}{\partial x^T} \right] \]

\[ W(t_0) = 0 \]

The Pontryagin Maximum Principle

Pontryagin Problem with Inequality Constraints
EXAMPLE:

\[ H = \lambda^T (A \lambda + B \mu) \]

\[ \mu \parallel \lambda \leq 1 \]

Optimum when \[ x^T B x \] occurs when \[ x = \frac{A}{2} \mu + \frac{B}{2} \lambda \]

\[ \lambda = \frac{1}{2} \]

\[ \mu = \frac{1}{2} \]
Admissible Trajectories:

Solutions following these constraints are:

\[ 0 \leq x(t) \leq a, \quad 0 \leq x^2(t) \leq a \]

If the car only goes forward:

\[ x^2(t) = 0 \]

These points are:

\[ x^2(t) = 0 \]

The state variable constraints:

\[ N \leq B(t) \leq 0 \]

Input Constraints:

\[ 0 \leq (t)^2 \leq 5 \]

If \( t \) is in phase variable form:

\[ \begin{bmatrix}
    0 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0
\end{bmatrix} = A 
\]

When a matrix is of the form:

\[ \begin{bmatrix}
    x^2 \\
    x \times 0 \\
    0 \\
\end{bmatrix} = \begin{bmatrix}
    1 & 1 & 1 \\
    0 & 0 & 0 \\
\end{bmatrix} + \begin{bmatrix}
    2 \\
    x^2 \\
    0 \\
\end{bmatrix} = \begin{bmatrix}
    x^2 \\
    x \times 0 \\
    0 \\
\end{bmatrix} = \begin{bmatrix}
    x^2 \\
    x \times 0 \\
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
\end{bmatrix} = 0 
\]

In normal form, \( x = \frac{d}{dt} x, \frac{d}{dt} x = \frac{d}{dt} x \) = \( \frac{d}{dt} x \). An example control problem:
\[ A(t) = A(t) + B(t) u(t) \]

**Nonlinear State-Variable System Representation**

**Has the following properties:**

- The **state transition matrix**
  \[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]

**Homogeneous Solution** \( u = 0 \)

- Linear: \( x(t) = A(t)x(t) + B(t)u(t) \)
- Nonlinear: \( x(t) = A(t)x(t) + B(t)u(t) \)
Solution is closed form

\[
\begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}
\Rightarrow
p(t) = \phi
\]

Then

\[
p(t) = e^{At}p
\]

(3) Eigen value approach

\[
(\Phi(t) = e^{At})
\]

\[
[1, 1][p] = [p][A]
\]

\[
A^P = P\]

Diagonalize A matrix

(3) Eigen value approach

\[
[\lambda_1 \lambda_2 \ldots \lambda_n]
\]

\[
\Phi(t) = \sum_{i=1}^{n} \lambda_i^t
\]

\[
X(s) = [S - A]^{-1}X(0) + \sum_{i=1}^{n} \lambda_i^t [e_i] X(0)
\]

(4) \( x(t) = A x(t) + B u(t) \)

Solution for time invariant case

(4) \( x(t) = A x(t) \)

(5) \( x(t) = \Omega x(t) \)
If \( A E X = M.0 \), then:

\[
\left( \int_{t_0}^{t_1} f(t) \, dt \right)_X = 0.
\]

Now, let \( M(t, t) \) be non-singular. Then:

\[
X(t_0) = X(t_1).
\]

Let:

\[
0 \neq \mathcal{U}(t_0) = \mathcal{U}(t_1).
\]

Proof: (Sufficiency)

1. Is a Necessary and Sufficient Condition

\[
L_x = \left( \int_{t_0}^{t_1} (f_x + f_t) \, dt \right) X = \left( \int_{t_0}^{t_1} f_t \, dt \right) X = X(t_0) - X(t_1) + \left( \int_{t_0}^{t_1} f_t \, dt \right) X = \left( \int_{t_0}^{t_1} f_t \, dt \right) X.
\]

Let:

\[
\mathcal{U}(t_0) = \left( \int_{t_0}^{t_1} f_t \, dt \right) \mathcal{U}(t_0) + \left( \int_{t_0}^{t_1} f_t \, dt \right) \mathcal{U}(t_1).
\]

As follows:

1. Is non-singular, these quantities:

\[
M(t_0, t_1) = \left( \int_{t_0}^{t_1} f(t) \, dt \right)_X + \mathcal{U}(t_0) = \mathcal{U}(t_1).
\]

Let:

\[
X = A(t) X(t_0) + B(t) \mathcal{U}(t).
\]

Theorem: The system:

\[
x(t_f) \text{ in finite time } t_f - t_0 \leq c,
\]

\[
eq 0 \text{ } x(t_f) \text{ can be transferred to } 0 \text{ if } x(t_0) = 0 \text{ and } x(t_0) \in \mathcal{U}(t_0) \text{ and } x(t_f) \in \mathcal{U}(t_f).
\]

A system is completely state controllable if:

\[
x = \int_{t_0}^{t_f} x(t) \, dt \text{ or } x(t) = x_0.
\]

Controllability
ASSUMPTION THAT \( C \neq 0 \) 

THIS CONTROLLABILITY TEST IS USEFUL 

\[ C^T C = 0 \iff C^2 = C = \ldots = C^n = 0 \]

\[
0 = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n \\
\end{bmatrix} = \begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n \\
\end{bmatrix} 
\]

\[ C = H_{2n}(t) \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix} \]

\[ C = H_{2n}(t) \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\end{bmatrix} \]

For controllability, the must be true: 

\[ \forall (t) = C \text{ (and, woos, } x(t_0) = 0) \]

\[ x(t) = \phi(t, t_0) x(t_0) \]

\[ x(t) = \phi(t, t_0) x(t_0) \]

WE HAVE 

\[ 0 = C^T M^o C \]

\[ 0 = C^T M^o C \]

\[ \text{Note that } M = M^o \iff C^T M^o C = 0 \]

\[ \text{Note that } M = M^o \iff C^T M^o C = 0 \]

\[ M^o = C \neq 0 \]

\[ M^o = C \neq 0 \]

\[ M^o = C \neq 0 \]

If state controllable, then \( A \in \mathbb{C} \) is singular for system.

Assume \( M^o (t_0, t) \) is singular. If the system 

PROOF OF CONTROLLATION

(MECESSARY)
IF $\text{No (or } M_0\text{)}$ is non-singular, then the column vectors of $H_T(t,\tau)$ are linearly independent. Then $H_T(t,\tau)$ is nonsquare, then

$H_Y(t,\tau)dt$ is nonsingular, then the system is completely output controllable on the time interval $[t_0, T]$. This is the same type of formulation we had before.

Let $Y_0 = C \int_{t_0}^{t} \hat{X}(t, \tau) \beta(t) dt$

$Y = Y_0 + \int_{t_0}^{t} \hat{Y}(t, \tau) \beta(t) dt$

$Y(t, \tau) = C \int_{t_0}^{t} \hat{X}(t, \tau) \beta(t) dt$

$X(t) = \phi(t, t_0) X_0 + \int_{t_0}^{t} \phi(t, \tau) \beta(\tau) d\tau$
Simple Test for Controllability

\[ x = Ax + Bu \]
\[ y = Cx \]

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]
\[ y(t) = Cx(t) \]

Consider a non-controllability situation.

Now, we know that if \( A(t) \) is not non-singular, then \( \dot{x}(t) \) is not non-singular.

Let \( \dot{x}(t) = 0 \) and \( \dot{y}(t) = 0 \).

Differentiating w.r.t. \( t \), we have:

\[ \dot{B}(t) = -A(t)B(t) + B(t)A(t)B(t) \]
\[ \dot{A}(t) = A(t)B(t) - B(t)A(t) \]

Thus, \( A(t) \) is not non-singular.

We have established that the system is controllable if all columns of \( A(\cdot) \) are non-singular.
(STATE) CONTROLLABILITY.

THE INEQUALITY NECESSARY FOR
WHERE WE HAVE ONE AGAIN INTRODUCED

\[ u^T \Phi(t, T) \int_0^1 \cdots \int_0^1 \Phi(t, T) = 0 \]

A SINGLE EQUATION GIVES

COMBINING ALL OF THESE RELATIONS INTO

\[ f^{i+1}(\tau) = f^i(\tau) - A(\tau) f^i(\tau) \]

WHERE

\[ f^i(\tau) = f^{i-1}(\tau) - A(\tau) f^{i-1}(\tau) \]

\[ u_1 \Phi(t, T)^{i-1} f^{i-1}(\tau) = 0 \]

IN GENERAL, WE HAVE THE RECURRENCE:

The n-l st derivative is

From (3) and (3), we have set a pattern.
A real (called a) is true if $||g||=0$.

The norm $||g||=0$, where $||g||=0$ if $g=0$.

A norm, $||g||$, must satisfy

- $0 \leq \text{the space} \leq \text{a real number}$
- Correspondence assigning each point

Space, the norm is a rule of

- Norm: In $\mathbb{R}^n$ in $\mathbb{R}^n$

Euclidean $\| f \| = \sqrt{f(\mathbf{a})^2 + \cdots + f(\mathbf{b})^2}$

Linear: $f(\cdot)$ is linear if $f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b})$.

Numbers are the Rationals

The functional and the associated

Real number, $x$ is the domain of

Each functional $f$ is a mapping assigning

- The domain, $\mathcal{D}$
- The range, $\mathcal{R}$

To $\mathbf{a}$ unique element in a set $\mathcal{A}$, $\mathcal{A}\subseteq \mathcal{D}$

- A Fundamental Concept
- A Fundamental Concept

A Fundamental Concept

A Fundamental Concept
A functional $F$ is given by $F[g] = F(g(x))$. The variation of a functional, $\delta F = \int (g(x) + \delta g(x))(g(x) + \delta g(x)) dx$, is obtained by expanding $F$ to a Taylor series. The first order terms in $\delta g(x)$ give the functional derivative $F'[g(x)]$. The extrema of a functional, $F[g]$, occur when $\delta F = 0$. The necessary condition for an extremum is $F'[g(x)] = 0$. For a global extremum, $F'[g(x)]$ must be a minimum or maximum at $g(x)$. The variation of $F$ is zero only if these conditions are fulfilled. Therefore, the variations of $F$ are unaltered if the only terms retained are linear in $\delta g(x)$. By expanding $F$ in a Taylor series and retaining only the first order terms, the variation on $\delta F$ gives the functional derivative $F'[g(x)]$. This is the variation of $F$ on $g(x)$. The variation of $F$ on $g(x)$ is $\delta F = \int \delta g(x) F'[g(x)] dx$. The variation of the functional is $\int (g(x) + \delta g(x))(g(x) + \delta g(x)) dx = \int \delta g(x) F'[g(x)] dx$. This is the Euler-Lagrange equation. The variations of a functional are unaltered if the extrema are global. The variations of $F$ are unaltered if the extrema are local.
For all $x$-omissible $x$:

$$6 \mid (x^* - x)$$

If $x$ * extremizes , then

$6$ any bounded, then

If $x$ is not constrained,

where $(x^* \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$)

Calculations of variations

Fundamental theorem of the

$\forall (x^*)$ is global extremum.

If condition holds $A$, then

$$A = \frac{\partial}{\partial (x^*)} - \int (x) - (x^*) d \text{min}$$

$$A = \frac{\partial}{\partial (x^*)} - \int (x) - (x^*) d \text{max}$$

$$A = \frac{\partial}{\partial (x^*)} - \int (x) - (x^*) d \text{min}$$

$A \in \mathbb{R}^n$, $\|x^* - x\| < \epsilon$

If extremum $x \to x + \epsilon > 0$

$\frac{\partial}{\partial (x^*)} - \int (x) - (x^*) d \text{max}$
\[ \begin{align*}
61 \pi & = \int_0^\infty \left\{ \frac{\sin x}{x^2 + \frac{\pi^2}{4}} + \frac{\sin x}{x^2 + \frac{3\pi^2}{4}} \right\} \ dx \\
\text{AND} \\
2 \int \left( e^{\frac{x^2}{2}} - \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} \right) dx & = 0 \\
\sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} & = \frac{\pi}{2} \int_0^\infty \frac{\sin x}{x^2 + \frac{\pi^2}{4}} \ dx \\
\int_0^\infty & = \sqrt{\pi} \\
\text{EXPANDING IN A TAYLOR SERIES:} \\
2 \int (x, t) B(x, t) & = f_x^2 + x^2 + x^3 + x^4 \\
\int (x, t) & = -c (x + t) \\
\text{NOW} \\
x_0 = x(t_0) \quad x(t) = x(t_0) + \dot{x} (\tau) \\
(6) \text{ TO AND IF ARE FIXED.} \\
(4) \text{G} \text{ IS TWICE DIFFERENTIABLE.} \\
\text{THIS FUNCTION GIVEN.} \\
\text{WE WISH TO FIND} \ x \text{ TO EXTREIMIZE} \\
2 \int \left( x (t) \right) \ dx \left( t \right) & = \int_0^\infty \left( x (t) \right) \ dx (t) \\
L \text{ THE SIMPLER PROBLEM.} \\
\text{\&} \\
\text{FUNCTIONALS OF A SINGLE FUNCTION.}
Then \( y(t) = 0 \) on

\[ \int_{t_1}^{t_2} \frac{dp}{\frac{x^2}{y^2}} \]

where \( \frac{dp}{\frac{x^2}{y^2}} \) is continuous on \([t_1, t_2]\). 

Calculus of Variations

Fundamental Lemma of the Calculus of Variations

We apply the Fundamental Theorem of Calculus for Variational Calculus to write

\[ 6x = \frac{d}{dt} \left[ \int_{t_1}^{t_2} \left( \frac{dx}{dt} \right)^2 - \frac{x^2}{y^2} \right] + \int_{t_1}^{t_2} \frac{dx}{dt} \frac{dx}{dt} - \frac{x^2}{y^2} \]

Since \( t_0 \) and \( t_1 \) are fixed,

\[ 6x = \frac{d}{dt} \left[ \int_{t_0}^{t} \left( \frac{dx}{dt} \right)^2 - \frac{x^2}{y^2} \right] + \int_{t_0}^{t} \frac{dx}{dt} \frac{dx}{dt} - \frac{x^2}{y^2} \]

Next, integrate by parts. Note that

\[ 6x(t) = 6x(t) \int_{t_0}^{t} \frac{dx}{dt} \frac{dx}{dt} - \frac{x^2}{y^2} \]
"Split Boundart Condition"

Are fixed. (Note the case where the end points optimal solution for the satisfies this is our.

The value of $x = x^*$ which

\[ \frac{dx}{dt} + \frac{x^3}{6} \Delta \frac{S}{a} = 0 \]

\[ E = \text{Equation} \]

Reduces to Eulera

Thus, our extrema problem
This is a natural boundary condition.

\[ 0 = \int_{\mathcal{S}} \frac{\partial}{\partial \mathbf{n}} = \int_{\mathcal{S}} \frac{\partial}{\partial \mathbf{n}} \mathbf{E} = \int_{\mathcal{S}} \frac{\partial}{\partial \mathbf{n}} \mathbf{D} = \mathbf{D} \cdot \mathbf{n} \]

Since \( \mathbf{D} \cdot \mathbf{n} \leq 0 \) is fixed, this
\[
\frac{\partial}{\partial \mathbf{n}} = 0
\]

Since \( \mathbf{D} \cdot \mathbf{n} \leq 0 \) is arbitrary,

\[
6 \mathbf{x} = 0
\]

\[
\mathbf{D} \cdot \mathbf{n} = 0
\]

We must satisfy the satisfying Euler's equation.

Thus, in addition to \( x(t_0) = 0 \), the solution of this equation

\[
\begin{align*}
\frac{\partial}{\partial t} & = 0 \\
\mathbf{E} & = \mathbf{D} \cdot \mathbf{n} \\
\mathbf{D} & = 0 \\
\mathbf{n} & = 0
\end{align*}
\]

of the previous section.

The integration by parts section.

To solve this case, we recall

\[
2. \text{ if specified, } x(t) \text{ free}
\]
0 = {\frac{d}{dx}} \left[ x \cdot \frac{e^x}{x^3} \right]

This reduces to

Now, since \( x_0 = x(0) \) is specified,

\[
0 = \frac{d}{dx} \left[ x \cdot \frac{e^x}{x^3} \right] = \frac{e^x}{x^3} + \frac{e^x}{x^2} - \frac{2xe^x}{x^3}
\]

Plus the condition

\[
0 = \frac{x}{3} + \frac{2e^{x}}{x} - \frac{e^{x}}{x^3}
\]

Euler's equation.

So, as before, we must satisfy

\[
\frac{d}{dx} \left[ x \cdot \frac{e^x}{x^3} \right] = \frac{e^x}{x^3} + \frac{e^x}{x^2} - \frac{2xe^x}{x^3}
\]

Was given after parts integration.

In our last section, the result is

2. \( t \neq 0, x_0 \) specified. \( x(0) \) free.
Thus, we may approximate:

\[ x \approx x(t_0) + \int_{t_0}^{t} \left( x'(t) \right) dt \]

Recognizing that \( x(t_0) = 0 \), secondary integrate by parts (as

First, approximate:

\[ \int_{t_0}^{t} x' \left( x(t') \right. dt' \approx x(t) \int_{t_0}^{t} x' \left( x(t') \right. dt' \]

Now

\[ x(t) \approx x(t_0) + \int_{t_0}^{t} \left( x'(t) \right) dt \]

With \( t \) and \( x(t) \) here:

\[ \int_{t_0}^{t} x' \left( x(t') \right. dt' = x \]

We still wish to minimize

3. Both \( t \) and \( x(t) \) free
Considered Two Pages A60

This Is The Same Case

\[ \frac{x^2}{2} \cdot \frac{P}{x} - \frac{x^2}{2} \cdot \frac{Q}{x} = 0 \]

And Here, \( G_x = 0 \)

Interpretations

Substituting This & Collecting Terms:

\[ \frac{f_x(x)}{f_x} \cdot \frac{x^2}{2} = \frac{x^2}{2} - x \left( \cdot f_x \right) \]

Thus, \( 5 \cdot (x^2) f_x = 6 \cdot f_x - f_x \cdot f_x \)

\[ f_x(x) \cdot \frac{x^2}{2} \cdot \frac{p}{x} - \frac{x^2}{2} \cdot \frac{q}{x} = 0 \]

To be continued on the next page.
\[ 0 = \frac{8}{b} + \left[ \left( \frac{8}{b} \right) - \frac{2h}{(h)(d)} \right] \frac{4}{3} \left( \frac{2}{3} \right) \]

Let us use:

Substituting & simplifying

Then

\[ x \left( \frac{2}{3} \right) = \left( \frac{5}{2} \right) \]

\[ x = \frac{5}{2} \]

\[ \text{And, } x \text{ is } \theta \text{ relative.} \]

\[ \theta = \frac{6}{b} \times x \]

\[ 0 = \frac{4}{3} \times \frac{6}{b} \times x \]

\[ 0 = \frac{8}{b} \times x \]

Specifically:

The relations in (a) & (b)

Here, we must satisfy all of

C, \( f + x \left( \frac{2}{3} \right) \) unspecified and independent

\[ \text{And, } \frac{6}{b} \times x = \frac{4}{3} \times \frac{6}{b} \times x \]

Thus, we must satisfy

\[ 4 \times \left( \frac{x}{b} \right)^2 - \left( \frac{x}{b} \right)^3 - \frac{4}{3} \left( \frac{x}{b} \right)^2 \left( \frac{x}{b} \right)^2 + \frac{4}{3} \left( \frac{x}{b} \right) \left( \frac{x}{b} \right)^2 - \frac{4}{3} \left( \frac{x}{b} \right)^3 = 0 \]

Here, \( x = \frac{4}{3} \) and

\[ b \left( \frac{2}{3} \right) \] specified.
\[
\begin{align*}
\frac{d}{dx} \left[ \frac{x}{x-1} \right] &= \frac{-x}{x-1}^2 - \frac{x^2}{x-1}^3 - \frac{x^3}{x-1}^4 + \frac{x^4}{x-1}^5 \\
\frac{d}{dx} \left[ \frac{x}{x^2-1} \right] &= \frac{-x^2}{x^2-1}^3 - \frac{x^3}{x^2-1}^4 \\
&+ \frac{x^4}{x^2-1}^5 \\
&\text{and} \\
0 &= \frac{dx}{dt} \left( \frac{1}{x} \right) - \frac{dx}{dt} \left( \frac{1}{x^2} \right) \\
\text{Our equations become}
\end{align*}
\]
\[ 0 = 2p \left[ x^{\frac{1}{2}} + \frac{x}{x + \frac{1}{2}} - \frac{x}{x + \frac{5}{3}} \right] + \left[ x^{\frac{1}{4}} - \frac{x}{x + \frac{1}{3}} \right] + \left[ x^{\frac{1}{6}} - \frac{x}{x + \frac{1}{3}} \right] \]

The transversality conditions are satisfied on both intervals.

Euler's equation must be

\[ \left[ 2p \left( \frac{x}{x + \frac{1}{3}} \right) - 1 \right] + \left[ \frac{x}{x + \frac{1}{3}} \right] = 0 \]

Now let

\[ J = \int (x) + \int (x') \]

Assume to \( t \) \( x_0 + x_4 \) are fixed.
\[ \frac{1}{2} \left[ x \left( \frac{x^3}{3} - \frac{3}{2} \right) \right] - \frac{5}{8} = \frac{1}{2} \left[ x \left( \frac{x^3}{3} \right) - \frac{5}{2} \right] \\
+ \frac{1}{2} \left( \frac{3}{3} \right) = \frac{-1}{2} \left( \frac{9}{3} \right) \]

Conditions are:
Weierstrass-Edman Corner

(1) For many functions, the

\[ \frac{1}{2} \left[ x \left( \frac{x^3}{3} - \frac{3}{2} \right) \right] + \frac{1}{2} \left[ x - \frac{3}{6 \sqrt[3]{\theta}} \right] + \frac{1}{2} \left[ \frac{6 \sqrt[3]{\theta}}{3} \right] = \]

\[ - \frac{1}{2} \left( \frac{6 \sqrt[3]{\theta}}{3} \right) + \frac{1}{2} \left[ x - \frac{3}{6 \sqrt[3]{\theta}} \right] + \frac{1}{2} \left( \frac{6 \sqrt[3]{\theta}}{3} \right) \]

Thus, we have

\[ 6 \sqrt[3]{\theta} = \frac{3}{6 \sqrt[3]{\theta}} \\
\theta^t = \frac{3}{6 \sqrt[3]{\theta}} = \frac{(3 \theta^t)}{6 \sqrt[3]{\theta}} \]

When \( x(t) = (t+\theta) \)

(2) When \( t \neq f \), \( x(t) \) is independent

\[ \frac{1}{2} \left[ x \left( \frac{x^3}{3} - \frac{3}{2} \right) \right] - \frac{5}{8} = \frac{1}{2} \left[ x \left( \frac{x^3}{3} \right) - \frac{5}{2} \right] \\
+ \frac{1}{2} \left( \frac{3}{3} \right) = \frac{-1}{2} \left( \frac{9}{3} \right) \]

Conditions are:
Weierstrass-Edman Corner

(3) When \( t \neq f \), \( x(t) \) is independent
Solution is
we have 2 eqns. + 3 unknowns,
2y' + p = 0 and 2y + p = 0
we must also require that
1 + y^2 - 5 is a constant hence
p \left( x^2 + p \right) = 0 = \frac{dy}{dp} \cdot \frac{dx}{dy} + \frac{dy}{dy} \cdot \frac{dx}{dx} + \frac{dy}{dx} \cdot \frac{dx}{dx} + \frac{dx}{dx} \cdot \frac{dx}{dx}
\begin{align*}
p &= \text{Lagrange Multiplier} \\
\frac{dy}{dx} &= \frac{y^2 + \frac{p}{2}(y^2 - 5)}{2y + p}
\end{align*}

Our augmented \( f \) is
subject to \( y_1^2 + y_2^2 = 5 \)
minimize \( f(y_1) = y_1^2 + y_2^2 \)

Example: Scalar use of Lagrange multipliers
\text{E. Constrained Extrema}
SOLVING GIVES

EXAMPLE: MINIMIZE

\[ f(y_1, y_2, \ldots, y_{n+m}) = 0 \]

subject to

\[ \begin{align*}
  y_1 + y_2 + y_3 &= 0 \\
  y_1 y_2 + y_2 y_3 &= \frac{1}{3} \\
  y_1 + y_2 + y_3 &= 1 \\
  y_1^2 + y_2^2 + y_3^2 &= 2
\end{align*} \]

Setting \( \frac{\partial f}{\partial y_i} = 0 \) for \( i = 1, 2, \ldots, n \), we get the equations

\[ \begin{align*}
  y_1 + y_2 + y_3 &= 0 \\
  y_1 y_2 + y_2 y_3 &= \frac{1}{3} \\
  y_1 + y_2 + y_3 &= 1 \\
  y_1^2 + y_2^2 + y_3^2 &= 2
\end{align*} \]

Then

\[ \begin{align*}
  \sum_{k=1}^{n+m} \frac{\partial f}{\partial y_i} y_i &= 0 \\
  \sum_{k=1}^{n+m} \frac{\partial f}{\partial y_i} y_i &= 0 \\
  \sum_{k=1}^{n+m} \frac{\partial f}{\partial y_i} y_i &= 0 \\
  \sum_{k=1}^{n+m} \frac{\partial f}{\partial y_i} y_i &= 0
\end{align*} \]

subject to

\[ \begin{align*}
  y_1 + y_2 + y_3 &= 0 \\
  y_1 y_2 + y_2 y_3 &= \frac{1}{3} \\
  y_1 + y_2 + y_3 &= 1 \\
  y_1^2 + y_2^2 + y_3^2 &= 2
\end{align*} \]
Note that

\[
\begin{pmatrix}
\vdots \\
\vdots
\end{pmatrix}
\]

is Lagrange multipliers.

Here, \( p(t) \) is an n-vector of

\[
\frac{\partial}{\partial t} \left( \frac{m}{2} \dot{w}(t)^2 + \frac{1}{2} \int_{0}^{t} p(t) \left( \frac{m}{2} \dot{w}(t')^2 + \frac{1}{2} \int_{0}^{t'} p(t'') \left( \frac{m}{2} \dot{w}(t'')^2 + \frac{1}{2} \int_{0}^{t''} \cdots \right) \right) \right) \]

is the augmented functional form of the newtonian equations, are therefore independent.

These \( \dot{w}, \ddot{w}, \ldots \) are point constraints (point constraints) to subject to

\[
\min \{ \int_{a}^{b} \left( w(t) \right) ^2 dt \} \quad \text{subject to} \quad \dot{w}(t) = 0, \quad \dot{w}(t) = \frac{\partial}{\partial t} \quad \text{for} \quad t = a, b, \ldots
\]
The whole mess gets simplified. That is, we could have simplified which is akin to Euler's Eq.

\[ 0 = \frac{m_g}{g} \left( \frac{p_e}{p} \right)^2 - \frac{m_g}{g} \left( \frac{p_e}{p} \right) + \frac{m_g}{2g} \]

Then this constraint becomes

\[ \tilde{\phi} = \phi + p \]

Augmented \( \phi \) as

Note that if we specify our

\[ 0 = \left[ \frac{p}{\frac{m_g}{g}} \left( \frac{p_e}{p} \right)^2 + \frac{m_g}{g} \left( \frac{p_e}{p} \right) - \frac{m_g}{g} \left( \frac{p_e}{p} \right) + \frac{m_g}{2g} \right] \]

A straightforward. Our other equation is:

Choose our Lagrange multipliers

First off, \( f'(W, \mu, t) = 0 \). We can

Boundary conditions

looks like we've specified all

\[ \tilde{\phi} = \phi + p \]

Terms containing \( \phi \). The result is:

We now integrate at parts.
5 EQUATIONS

\[ \begin{align*}
   \mu_1' - \pi_1' - 2\mu_2' &= 0 \\
   \mu_2' &= -2\mu_2 - \mu_3' \\
   \mu_3' &= -2\mu_4 - \lambda_1 - \lambda_3 \\
   \lambda_1 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_2 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_3 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_4 &= -2\lambda_1 - \lambda_3 - \lambda_4 \\
\end{align*} \]

THE CONSTRAINTS

UTILIZE THESE EQUATIONS WITH

\[ \begin{align*}
   \mu_1 &= x_1 - x_2 + u' \\
   \mu_2 &= x_1 - x_2 + u' \\
   \mu_3 &= x_1 - x_2 + u' \\
   \lambda_1 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_2 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_3 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_4 &= -2\lambda_1 - \lambda_3 - \lambda_4 \\
\end{align*} \]

EXAMPLE

MINIMIZE

\[ J(x, u) = \int_{t_0}^{t_1} \left[ x_2 + x_2 + u \right] dt \]

SUBJECT TO:

\[ x_1' = x_2 - x_2 + u \]

\[ u \leq u' \leq 0 \]

3 EQUATIONS WITH 3 UNKNOWN

\[ \begin{align*}
   \mu_1 &= x_2 - x_2 + u' \\
   \mu_2 &= x_2 - x_2 + u' \\
   \mu_3 &= x_2 - x_2 + u' \\
   \lambda_1 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_2 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_3 &= -2\lambda_2 + \lambda_3 - \lambda_4 \\
   \lambda_4 &= -2\lambda_1 - \lambda_3 - \lambda_4 \\
\end{align*} \]

MINIMIZE

\[ J(x, u) = \int_{t_0}^{t_1} \left[ x_2 + x_2 + u \right] dt \]

SUBJECT TO:

\[ x_1' = x_2 - x_2 + u \]

\[ u \leq u' \leq 0 \]

THEORY
The Lagrange multipliers are always constants.

\[ p(t) = 0 \]

Therefore, the equation is thus always

\[ \frac{\partial Q}{\partial x} - p(t) \]

and

\[ \frac{\partial Q}{\partial y} - p(t) \]

Note that

\[ \frac{\partial^2}{\partial x^2} \psi = 0 \]

This gives the master equation

\[ \frac{\partial}{\partial t} \right( \psi(t, x, y) \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} \right( \psi(t, x, y) \frac{\partial}{\partial x} \right) \]

Plus, the constraints:

\[ 0 = \frac{\partial}{\partial x} \right( \psi(t, x, y) \frac{\partial}{\partial x} \right) \]

\[ 0 = \frac{\partial}{\partial y} \right( \psi(t, x, y) \frac{\partial}{\partial y} \right) \]

Plus, we still have the master equations:

\[ 0 = \frac{\partial}{\partial t} \right( \psi(t, x, y) \frac{\partial}{\partial x} \right) \]

\[ 0 = \frac{\partial}{\partial t} \right( \psi(t, x, y) \frac{\partial}{\partial y} \right) \]

Solving all these gives us the equations:

Define

\[ \psi(t, x, y) = e^{(x, y, t)} \]

Now:

\[ Z(t) = \int_{C(t)} e^{(x, y, t)} dx dy dt \]

The constraints are specified as:

\[ \int_{C(t)} e^{(x, y, t)} dx dt = \mathcal{C} \]

Contraints are of the form:

\[ \int_{C(t)} e^{(x, y, t)} dx dt = \mathcal{C} \]

\[ \int_{C(t)} e^{(x, y, t)} dx dt = \mathcal{C} \]
\[ z(t_0) = 0 \neq c \]

And the boundary conditions
\[ z = w^2 \]

Also, we have constraints
\( F(t) \leq 0 \)
\( F(t) = 0 \)
\( g_1 \leq \frac{\partial}{\partial t} = 0 \)
\( g_2 = v \frac{\partial}{\partial t} + w \frac{\partial}{\partial x} + \frac{\partial}{\partial w} \]
\( g_3 = \frac{\partial}{\partial t} \]
\( g_4 = \frac{\partial}{\partial t} \]
\( g_5 = \frac{\partial}{\partial t} \]
\( g_6 = \frac{\partial}{\partial t} \]

Now,
\[ z = w^2 \]

Subject to
\[ \int_{t_0}^{t_f} \sqrt{v^2 + w^2 + z^2} \]
In order to solve, we must solve the following:

\[
\begin{align*}
x &= f(t) + \mathbf{u} = \mathbf{f}(t) + \mathbf{u} \\
x &= -5x/5x \\
\end{align*}
\]

Put bounds on control:

\[
\begin{align*}
0 &\leq (h^T + 2\mathbf{c})^T \mathbf{x} + (h^T + 2\mathbf{c}) \\
\mathbf{A} &\geq 0 \\
\mathbf{x}^T &\geq 0 \\
\mathbf{e}^T &\geq 0 \\
\mathbf{e}^T + \mathbf{c} &\geq 0 \\
\end{align*}
\]

Use P's minimum principle:

\[
\mathbf{x} = \phi + \mathbf{f}(t) + \mathbf{u} + \mathbf{c}
\]

Hamiltonian is:

\[
J = \phi^T \mathbf{x}(\mathbf{t}_f) + \int_{t_0}^{t_f} \frac{1}{2} \mathbf{x}^T \dot{\mathbf{x}} + \mathbf{c}(\mathbf{x}, \mathbf{t}) + \mathbf{f}(\mathbf{x}, \mathbf{t}) \, dt
\]

Differentiation w.r.t. \( \mathbf{u} \) assumed linear.
\[ s(t) = \int_0^t \left( b - a(t - 5) \right) dt \] if \( a < 0 \)

**Solution is**

\[ q = 0 \]
\[ q \in \mathbb{R} \]

**Then**

\[ X = q^T A \]
\[ \text{and} \]
\[ x(t) = x(0) x(t) = (x - \delta) \] \( \forall \) \( t \)

**Let**

\[ y(t) = t^2 - b \]

**Solve**

\[ x = x + b = x - b a \]
\[ x(0) = 0 \]

**Thus**

\[ q = x(0) \]
\[ q \in \mathbb{R} \]

**P. S. Minimum Principle is**

\[ x(t) = 1 + \lambda^T (A \lambda + b) \]
\[ t \geq 0 \]
\[ \lambda^T = 1 \]
\[ D = t \]
\[ = \int_0^t \frac{1}{t} dt = t \]
\[ x = A x + b \]

**Minimum Time Problem**
The four possible optimal controls are:

1. $u = \pm \frac{1}{2} t + C_3$
2. $u = \pm \frac{1}{2} t + C_4$
3. $u = \pm \frac{1}{2} t + C_3$
4. $u = \pm \frac{1}{2} t + C_4$

Now, $x_1 = \frac{1}{2} t + C_3$
$x_2 = \frac{1}{2} t + C_4$

For $u = 1$, use the sign $= x = \frac{1}{2} x_2 + C_3$
For $u = -1$, use the sign $= x = \frac{1}{2} x_2 + C_4$

Costate Equations:

$\lambda_1 = -s x_1 \Rightarrow \lambda_1 = c_1$
$\lambda_2 = -s x_2 \Rightarrow \lambda_2 = c_2$

Now $\dot{x}_1 = x_2$
$\dot{x}_2 = u$

Example

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1(t) = x_0$
$x_2(t) = t x_0$
IMPLEMENTATION:

\[ \dot{x_1} = \dot{x_2}, \quad \dot{x_2} = 0 \]

**Given:**

\[ x_1' = -\frac{x_2}{1+x_2^2} \]

As shown, the switching curve is a curve corresponding to different curves. The four cases allowed by the A M switching curve are:

1. Case A
2. Case B
3. Case C
4. Case D
THE NUMBER $x_m$

Special Polarity Conditions
Do so, we must consider some
We wish to minimize this expression.

$$\sum_{j=1}^{m} \max\{c_j, F_j(x)\}$$

Assuming $c_i \geq 1$.

Apply MIN PRINCIPLE Term-wise.

So:

$$x_m = \min \{x_j : j = 1, \ldots, m\}$$

Denote columns of $G$ at $g_{i,j}$.

$$[\ldots g_{1,m} \ldots g_{n,m} \ldots]$$

Now:

$$x_C \in \arg\min \{c_j, F_j(x)\}$$

Applying MINIMUM PRINCIPLE

$$x_C = \sum_{j=1}^{m} c_j + \min\{c_j, F_j(x)\}$$

$$x = \begin{cases} x + [c_j + F_j(x)]_j & \text{if } c_j \geq F_j(x) \\ 0 & \text{otherwise} \end{cases}$$

Minimum Fuel Problem
\[ u' > 0 \quad \text{for minimization (2)} \]

\[
\begin{cases}
\text{NO} & \text{for } u < 0 \\
\text{Hi} + \frac{1}{\sqrt{T}} u' = \frac{1}{2} u' & \text{for } u > 0 \\
\text{Hi} & \text{for } u = 0
\end{cases}
\]

To minimize, we would choose \( u \) as

\[ u' > 0 \quad \text{for } u > 0 \]

\[ u' = 0 \quad \text{for } u = 0 \]

\[ u' < 0 \quad \text{for } u < 0 \]

\[
\begin{align*}
\text{let } & 
\frac{1}{T} \ll \frac{1}{\sqrt{T}}
\end{align*}
\]
Singular points occur when $K' = 0$.

\[ x, x^2, 0, 0, 0 \] \[ x', x^2, 0, 0, 0 \]

Now if $0 \leq x \leq x^2 = \frac{5}{6}$

\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x^2 \end{bmatrix} = \begin{bmatrix} x \end{bmatrix} \]

For $u = 0$, we have to proceed. Now for $u = 1$, we have the same problem as before.

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} x \end{bmatrix} \]

Thus, $\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Here $\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore, $\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The situation turns out:

\[ \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ x = \sqrt{\frac{q}{1}} \]

Example
Our control is determined by \( y_2 \), which

\[
y_2 = 0, \quad y_2 = \alpha + t, \quad \alpha > 0
\]

Let \( \alpha = 0 \), for \( y_2 \)'s case.

We must now solve \( (\text{in general}) \) for \( y_2 \).
Solving for $a_1$:

From $0 = c_0 x^2 + c_1 x + c_2$, we get $x = x_0$.

Also, $a_1 = 0$, since $x = x_0$.

Moreover, $a_2 = 0$, since $x = x_0$.

Thus, $X(t) = k t + k^2$.

We wish to show that $t = -t_0$. 

Consider the contract $u = 1, t = 1$. 

\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} + 
\begin{bmatrix}
k \\
k^2
\end{bmatrix}
\]

where $X(0) = x(0)$ and $X'(0) = k^2 + k$.
At \( v = 0 \),

\[ t_2 - t_1 = \frac{a}{\frac{d}{2}} = \infty \]

This gives \( a = 0 \), but

It is not  its optimum trajectory must make

For \( u = 0 \), \( \dot{X} = \left[ \begin{array}{c} \dot{x} \\ \dot{v} \end{array} \right] = \left[ \begin{array}{c} 1 & 1 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{c} x_2 \\ v \end{array} \right] \)

For \( u = 0 \), \( J = 0 \)

\[ J = J_1 + J_2 (A X + B u) \]

This problem can be formulated
Thus, we have:

\[ x = 0 \iff x = 1 + c \]

Assume \( u = 1 \) and \( y > 0 \)

\[ x = u \iff x = -c + c \]

Assume \( u = 1 \) and \( y > 0 \)

\[ x(t) = 0 \]

\[ y(t) = \begin{cases} 0 & x \geq 1 \\ \frac{1}{2} & x < 1 \end{cases} \]

For \( x = 0 \), \( y = \frac{1}{2} x^2 \)

Then our Hamiltonian is

\[ H = \frac{1}{2} \int_0^T \left( x(t)^2 + 1 \right) dt \]

\[ \frac{dx}{dt} = \begin{cases} 0 & x \geq 1 \\ \frac{1}{2} x^2 + 1 \end{cases} \]

Example

\[ x = \begin{cases} 0 & y < \frac{1}{2} x \\ 1 - y & \frac{1}{2} x < y < 1 \\ 1 & y \geq 1 \end{cases} \]

\[ y = \begin{cases} -\frac{1}{2} x & x \geq 0 \\ 0 & x < 0 \end{cases} \]

The singular problem

\[ x = -x + b \]
Our solution is thus:

\[ y(t) = \begin{cases} 
  0 & t \leq 0 \\
  y(0) & t > 0 
\end{cases} \]

\[ y' = \begin{cases} 
  -x & t < 0 \\
  0 & t = 0 \\
  y' & t > 0 
\end{cases} \]

\[ y(0) = \begin{cases} 
  0 & t = 0 \\
  y(0) & t > 0 
\end{cases} \]

During the singularity

\[ x(t) = \begin{cases} 
  0 & t < 0 \\
  t \quad & t = 0 \\
  x & t > 0 
\end{cases} \]

\[ y(0) = \begin{cases} 
  0 & t < 0 \\
  y(0) & t = 0 \\
  y' & t > 0 
\end{cases} \]

For \( t = 0 \), we have
This is our singular AF: 
\[ Ax = x, \quad \lambda Ax = x \] 
\[ \lambda = 1, \quad x = 0 \]

Thus, from the state equations 
\[ x' = f(x, u), \quad y = g(x, u) \]

Now, from (8) \( v = x \) and \( x' = x \). 
Thus, \( y' = x \). 
When \( x = A^{-1}x = 0 \) or \( x = A - 1/2 \).

We have a singular solution 
\[ x(t) = \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{\sqrt{2}}} \\ \frac{t}{\sqrt{2}} \end{bmatrix} \]

\[ (\frac{t}{\sqrt{2}})^2 + x^2 + x | x = \frac{t}{\sqrt{2}} \]

\[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = A \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} \]

Example: \( J = \int x^2 \) 

What is \( C \)?

For singular problems, the optimal control is \( u(t) = 0 \)

\[ J = \int (x(t) + y(t)) \]
In terms of $x^2$, $t = \frac{x_0 - x_2}{x^2 - x_0}$ gives

\[
\begin{bmatrix}
X'_2(t)
\end{bmatrix} = \begin{bmatrix}
X_2(t)
\end{bmatrix}
\]

\[
x'_2 = x_2 \Rightarrow x_2 = \frac{\xi}{x_2 + \xi} \Rightarrow x_2 = K\]

Consider now limiting $\xi \to K$

\[
\text{Solution for } x^2 \leq K
\]

On the singular arc

\[
\frac{2}{x_1^2} = x_1
\]

Thus, $x_1 = x_2$, $x_1$ is not an explicit function

At $x_1 = x_2$, $x_1 = x_2$
We wish to avoid the vertical strip.

Let $a > 0$. Our singular arcs become

\[(x, x) \rightarrow (x, x) \quad (\text{Assume } k = 2)\]

Example

\[
\begin{align*}
\text{For large } k, \\
x_1 &= (x_2 + k^2) + x_10
\end{align*}
\]

For large $k$,

\[
x_1' = -\frac{k}{x_2 - x_20} \left( \frac{x_20 - x_10}{x_20 + k^2} \right) + \frac{1}{k} \left( \frac{x_20 + k^2}{x_20 - x_20} \right) x_10
\]
We can straightforwardly solve for $x(t)$ and $y(t)$ using the initial conditions:

$\begin{align*}
&x(t_0) = x_0 \\
v_x(t_0) = 0 \\
&y(t_0) = 0 \\
v_y(t_0) = v_0
\end{align*}$

Also:

$\begin{align*}
&x(t_f) = x_f \\
v_x(t_f) = v_x \\
&y(t_f) = y_f \\
v_y(t_f) = v_y
\end{align*}$

Using De Moivre's Theorem, we can express the solution:

$\phi = C_1 e^{i \omega t_f}$

Now:

$\begin{align*}
&x(t_f) = x_0 + \int_0^{t_f} v_x \, dt \\
v_x(t_f) = v_x \cos(\omega t_f)
\end{align*}$

The general solution would be:

$\begin{align*}
&x(t) = x_0 + \int_0^t v_x \cos(\omega t) \, dt \\
&y(t) = y_0 + \int_0^t v_y \sin(\omega t) \, dt
\end{align*}$

Consider homogeneous solutions:

These are obtained from Hamiltonian:

$\begin{align*}
&x(t_f) = x_f \\
v_x(t_f) = v_x \\
v_y(t_f) = v_y
\end{align*}$

Consider a linearization...
\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \]

STOP ITERATION WHEN
USE THEM TO FIND \( x(2) \) AND \( y(2) \), ETC.
WE CAN SOLVE FOR \( x(1) \) AND \( y(1) \)

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Note these are of the form

\[ \begin{bmatrix} \frac{x_0}{\frac{1}{2}} & 0 \\ 0 & \frac{x_0}{\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{x_0}{\frac{1}{2}} & 0 \\ 0 & \frac{x_0}{\frac{1}{2}} \end{bmatrix} \]

\[ \begin{bmatrix} \frac{x_0}{\frac{1}{2}} & 0 \\ 0 & \frac{x_0}{\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{x_0}{\frac{1}{2}} & 0 \\ 0 & \frac{x_0}{\frac{1}{2}} \end{bmatrix} \]

REARRANGING

\[ \begin{bmatrix} x_0 & 0 \\ 0 & x_0 \end{bmatrix} \]

Let \( x(0) \) and \( y(0) \) be an initial state.
\( \forall t \neq 0 \), \( x(t) \) and \( y(t) \) are in STATE COSTATE EQUATIONS

LINEARIZATION OF REDUCED
\[
\begin{align*}
\mathbf{0} = (0, 0, 0, 0) & \quad \Rightarrow \quad \mathbf{B} = \mathbf{A} - 2\mathbf{x} + \mathbf{z} = \mathbf{y} \\
\varepsilon = (0, \mathbf{0}) & \quad \Rightarrow \quad \mathbf{x} - \mathbf{z} = \mathbf{y} \\
\mathbf{y} = \mathbf{0} & \quad \Rightarrow \quad \mathbf{y} = \mathbf{x} + \mathbf{z} = \mathbf{0} \\
\mathbf{X} - \mathbf{Y} & \quad \Rightarrow \quad \mathbf{X} = \mathbf{Z} \\
\mathbf{X} + \mathbf{Z} & \quad \Rightarrow \quad \mathbf{X} = \mathbf{Z} \\
\mathbf{X} - \mathbf{Z} & \quad \Rightarrow \quad \mathbf{X} = \mathbf{Z} \\
\mathbf{X} & \quad \Rightarrow \quad \mathbf{x} = \mathbf{x} \\
\text{Example:} & \\
\mathbf{X} & = \mathbf{x} \\
\mathbf{Y} & = \mathbf{y} \\
\mathbf{Z} & = \mathbf{z}
\end{align*}
\]
\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
x_1 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\vdots \\
0 \\
x_1^p \\
\vdots \\
x_n^p
\end{bmatrix}
\]

where the G.C. of the particular solution are

\[
\begin{bmatrix}
x_1^p \\
\vdots \\
x_n^p
\end{bmatrix}
\]

The general solution is of the form:

\[
\begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\vdots \\
0 \\
x_1^p \\
\vdots \\
x_n^p
\end{bmatrix}
\]

The homogeneous boundary conditions are

\[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
x_1^p \\
\vdots \\
x_n^p
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\vdots \\
0 \\
x_1^p \\
\vdots \\
x_n^p
\end{bmatrix}
\]

The Strut-End-Frame Generalization is

Both \( x \) and \( A \) are now in vectors.
\[
\begin{bmatrix}
\vdots \\
C_0 \\
\vdots
\end{bmatrix}
\]
A square matrix

\[
A = \begin{bmatrix}
& & & & \lambda_1(t_f) \\
& & & \lambda_2(t_f) \\
& & \ddots & & \vdots \\
& \vdots & & \ddots & \lambda_n(t_f)
\end{bmatrix}
\]
Each entry is an n x n column.

\[
\begin{bmatrix}
\lambda_1(t_f) & \vdots & \lambda_n(t_f)
\end{bmatrix}
\]
\(\lambda(t_f) = C_0 + \lambda(t_f)
\)
THE METHOD OF STEEPST DESCENT

Consider the problem of minimizing

\[ f(x, y) = x^2 + y^2 \]

We wanna go in opposite direction of gradient:

\[ \nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \]

Define the unit vector:

\[ \hat{u} = \frac{\nabla f}{||\nabla f||} = \frac{1}{\sqrt{2x^2 + 2y^2}} \begin{bmatrix} 2x \\ 2y \end{bmatrix} \]

Then

\[ x_{n+1} = x_n - \gamma \hat{u} \]

or

\[ y_{n+1} = y_n - \gamma \hat{u} \]

for \( \gamma \) step size.
Equation only if \( \frac{50}{3} + \frac{50}{4} + \frac{50}{4} = 0 \)

\[
0 \leq \frac{50}{3} + \frac{50}{4} + \frac{50}{4} \rightarrow \text{then} \]

\[ s_0 = 0 \quad \text{if} \quad C(x) = -\frac{3}{4} \theta(x) \quad \text{if} \quad \theta > 0 \]

\[ s_0 = \theta(0) \quad \text{if} \quad \theta < 0 \]

\[ \text{Let} \]

\[ s = \frac{\theta}{4} + \frac{\theta}{4} + \frac{\theta}{4} \]

\[ s = \frac{\theta}{4} + \frac{\theta}{4} + \frac{\theta}{4} \]

\[ s = \frac{\theta}{4} + \frac{\theta}{4} + \frac{\theta}{4} \]

\[ \text{Assume these conditions hold, cut} \]

\[ \frac{e_x}{4} = \frac{e_x}{4} \]

\[ \text{Boundary Conditions:} \]

\[ \frac{e_x}{4} = \frac{e_x}{4} \]

\[ \frac{e_x}{4} = \frac{e_x}{4} \]

\[ \frac{e_x}{4} = \frac{e_x}{4} \]

\[ \frac{e_x}{4} = \frac{e_x}{4} \]

\[ \text{Problem is} \]

\[ \text{Steepest Descent} \]

\[ \text{Minimization of Functionals} \]
6. Compute

\[ u(0) = u(0) - \frac{\theta}{6} \]

7. Compute \( \theta \)

8. Evaluate \( \theta = 6 \)

9. Integrate \( \cos^4 \) (Eulerian)

\[ \int_0^{\infty} x \cdot e^{-x} \, dx = \frac{\theta}{6} \]

10. Form \( x(t) \)

11. Compute \( u(t) \)

12. Choose \( u(0) \)

Steepest Descent Algorithm
\[ \frac{d^2x}{dt^2} = f(x, \dot{x}) \]

Initial conditions:
\[ x(0) = x_0, \quad \dot{x}(0) = v_0 \]

Example:
\[ f(x, \dot{x}) = x^2 - x + 1 \]
\[ \frac{2p \cdot h}{\left( x - x_0 \right)^2 + h^2} = 0 \Rightarrow \]
\[ 2p \cdot \left( x - x_0 \right)^2 + h^2 = 2p \cdot \left( x - x_0 \right)^2 \]
\[ x = x_0 \]

**Example**

\[ \int_{x_0}^{x_0 + \Delta t} R \, dx = \int_{x_0}^{x_0 + \Delta t} f(t) \, dt \]

Thus, by Fundamental Lemma of

Thus, pick \( \Delta t \) to minimize \( R \).

Assume we know \( x \) and want to approximate \( x = f(x, y, t) \).
Our state equations are thus:

\[
\frac{dp}{dt} = \left[ \begin{array}{cc}
\frac{g}{b} & \frac{g}{b} \\
\frac{g}{b} & \frac{g}{b}
\end{array} \right] \frac{p_0 - \frac{50}{b}}{b} - \frac{p_0 - \frac{50}{b}}{b} = \left[ \begin{array}{c}
0 \\
0
\end{array} \right]
\]

Thus, the sum of the entering mass is the rate of change of salt in tank B.

The rate of change of salt in tank A:

Thus:

\[
\frac{dp}{dt} = \frac{50}{b} g \text{ mol/min}
\]

And:

\[
\frac{d}{dt} = \frac{V}{m} \text{ mol/min}
\]

Cut

\[
\frac{dp}{dt} \frac{\Delta P}{P} = \frac{\Delta P}{P}
\]

\[P(t) = 80 \text{ Pa}
\]

\[P(0) = 60 \text{ Pa}
\]
\[
\begin{bmatrix}
\frac{s^2 + \frac{1}{\tau}}{s + \frac{1}{\tau}} + \frac{1}{\tau} & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
0
\end{bmatrix} = \begin{bmatrix}
(+) & 0 \\
0 & (+) 
\end{bmatrix} \begin{bmatrix}
\theta' \\
0
\end{bmatrix} \\
\begin{bmatrix}
\frac{s}{\tau} + \frac{1}{\tau} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta' \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \\
\begin{bmatrix}
\frac{s}{\tau} + \frac{1}{\tau} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta' \\
0
\end{bmatrix} = 0
\]

\[d(t) = \begin{bmatrix}
0 \\
0
\end{bmatrix}\]

\[\begin{bmatrix}
\frac{s}{\tau} + \frac{1}{\tau} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta' \\
0
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \\
\begin{bmatrix}
\frac{s}{\tau} + \frac{1}{\tau} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta' \\
0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(c) Again, let's use Laplace:

\[\text{Which gives:}
\begin{bmatrix}
0 \\
0
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\frac{\theta}{\tau} & 0 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\frac{\theta}{\tau} & 0 \\
0 & 0
\end{bmatrix}
\]

\[\text{State Equations:}
\]

\[\text{Laplace Transforming The System as Linear Time Invariant.}
\]

(6) Since we have modeled the...
\( p(t) = \frac{5}{4} t e^{-\frac{5t}{8}} + \frac{5}{2} \) Pounds of Salt

\( q(t) = 60 t e^{-\frac{80t}{8}} \) Pounds of Salt

\( \frac{\partial}{\partial t} \begin{bmatrix} 0 \\ 0 \\ 60 \end{bmatrix} = \begin{bmatrix} e^{-\frac{80t}{8}} & 0 & 0 \\ 0 & e^{-\frac{80t}{8}} & 0 \\ 0 & 0 & e^{-\frac{80t}{8}} \end{bmatrix} \begin{bmatrix} -\frac{80t}{8} \\ 0 \\ 60 \end{bmatrix} \)

Setting \( t = 0 \), we thus have

\[ x(t) = (t - t_0) x(t_0) \]

System with no inputs

Now, for the linear time invariant

\[ \Phi(t) = \begin{bmatrix} e^{At} \\ 0 \\ 0 \end{bmatrix} \]

We have

\[ \Phi(t_1) \Phi(t_2) = \Phi(t_1 + t_2) \]

And

\[ x(t_1 + t_2) = \Phi(t_1) x(t_2) \]

Since
The signal flow graph is then

\[
\begin{bmatrix}
\frac{M}{s^2} & X_2 \\
0 & \frac{M}{s}
\end{bmatrix}
+ \begin{bmatrix}
X_1 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{M}{s} & \frac{M}{s} \\
-\frac{K}{s} & -\frac{K}{s}
\end{bmatrix}
= \begin{bmatrix}
\frac{X_1}{s^2} \\
\frac{X_2}{s}
\end{bmatrix}
\]

(b) Using Laplace transforms:

\[u(t) = f(t)\]

Where our control input is

\[
\begin{bmatrix}
\frac{M}{s^2} & X_2 \\
0 & \frac{M}{s}
\end{bmatrix}
+ \begin{bmatrix}
X_1 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{M}{s} & \frac{M}{s} \\
-\frac{K}{s} & -\frac{K}{s}
\end{bmatrix}
= \begin{bmatrix}
\frac{X_1}{s^2} \\
\frac{X_2}{s}
\end{bmatrix}
\]

Then

\[
\begin{align*}
x_1 &= y \\
x_2 &= \frac{M}{s} x_1 + \frac{M}{s} x_2 - \frac{K}{s} x_1 - \frac{K}{s} x_2
\end{align*}
\]

This force from Spring = \(\frac{M}{s}\) force from Vector = \(\frac{M}{s}\) force

(2) Force from Vector \(= M \frac{X_1}{s^2}\)

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\begin{bmatrix}
\frac{M}{s^2} & \frac{M}{s} \\
-\frac{K}{s} & -\frac{K}{s}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

(1 - \(\frac{M}{s}\))
If we wished to do the clock:

\[
\frac{W}{g} + s \frac{M}{X} = \frac{(W + s)M}{1 + g}
\]
\[ M = \frac{1}{k}, \quad K = \frac{1}{m}, \quad E = \frac{1}{\rho L} \]
\[
\begin{align*}
\left[\begin{array}{c}
\frac{t}{4} + e^- & e^+ \\
\frac{t}{4} & e^- - e^+
\end{array}\right]
&= \\
\left[\begin{array}{c}
\frac{t}{4} - e^- & e^+ \\
\frac{t}{4} & e^- - e^+
\end{array}\right]
\end{align*}
\]

\[
(F - I - A) I - F = I
\]
We must now evaluate these integrals.

\[
\int_{0}^{2\pi} e^{-t} \cos(t + \frac{\pi}{2}) \, dt = \left[ \frac{1}{2} e^{-t} \cos(t + \frac{\pi}{2}) \right]_{0}^{2\pi} \\
+ \int_{0}^{2\pi} \frac{1}{2} e^{-t} \sin(t + \frac{\pi}{2}) \, dt = \left[ -2 e^{-t} \sin(t + \frac{\pi}{2}) \right]_{0}^{2\pi} \\
+ \int_{0}^{2\pi} \frac{1}{2} e^{-t} \cos(t + \frac{\pi}{2}) \, dt = \left[ e^{-t} \cos(t + \frac{\pi}{2}) \right]_{0}^{2\pi}
\]

For \( y(t) = 0 \) and \( x(t) y(t) \geq 0 \),

\[
y(0) = \frac{1}{10} - 2e^{-t} \cos(t + \frac{\pi}{2})
\]
\[
\left( \frac{h}{2m} - \frac{\partial}{\partial t} \right) \times \phi(t) + \frac{1}{\sqrt{2}} \exp \left( \frac{i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t) + \frac{1}{\sqrt{2}} \exp \left( \frac{-i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t)
\]

Thus

\[
(\text{tr} - \text{tr} + \text{tr} - \text{tr}) - \frac{1}{\sqrt{2}} \exp \left( \frac{i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t) + \frac{1}{\sqrt{2}} \exp \left( \frac{-i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t)
\]

Thus

\[
(\text{tr} - \text{tr} + \text{tr} - \text{tr}) - \frac{1}{\sqrt{2}} \exp \left( \frac{i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t) + \frac{1}{\sqrt{2}} \exp \left( \frac{-i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t)
\]

Thus

\[
(\text{tr} - \text{tr} + \text{tr} - \text{tr}) - \frac{1}{\sqrt{2}} \exp \left( \frac{i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t) + \frac{1}{\sqrt{2}} \exp \left( \frac{-i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t)
\]

Thus

\[
(\text{tr} - \text{tr} + \text{tr} - \text{tr}) - \frac{1}{\sqrt{2}} \exp \left( \frac{i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t) + \frac{1}{\sqrt{2}} \exp \left( \frac{-i \theta}{\sqrt{2}} \right) \phi(t) = \phi(t)
\]

Thus
\[\begin{align*}
\phi(t) &= x^2(t) = -\frac{1}{4} e^{-\frac{t}{2}} + 2e^{-2t} + 2t e^{-2t} + \frac{1}{4} e^{-\frac{t}{2}} \cos \left(\frac{\sqrt{2} t}{2}\right) + \frac{1}{4} e^{-\frac{t}{2}} \sin \left(\frac{\sqrt{2} t}{2}\right) + 2t e^{-2t} + 2e^{-2t}.
\end{align*}\]

**Summary:**

Thus,

\[\begin{align*}
\phi(t) &= x(t) = -\frac{1}{4} e^{-\frac{t}{2}} + 2e^{-2t} + 2t e^{-2t}.
\end{align*}\]

**Since** \(\cos \left(\frac{\sqrt{2} t}{2}\right) = \cos \frac{\theta}{2} = \frac{1}{2} (e^{i\theta} + e^{-i\theta})\), we evaluate.

\[\begin{align*}
\phi(t) &= x(t) = -\frac{1}{4} e^{-\frac{t}{2}} + 2e^{-2t} + 2t e^{-2t}.
\end{align*}\]
Similarly, for Tank 2:

\[ \frac{dV_2}{dt} = \frac{A_2 h_2}{k_a - k_h + w_1(t) + w_2(t)} \]

Thus:

\[ V_2 = c_2 \]

Assume \( g(t) = k_1 (a - h_1) \)

The volume in Tank 1 is homogeneous.
November 22, 2017

Dr.

1. In matrix form

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
\dot{z}(t)
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta & 0 \\
\gamma & \delta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x(t) \\
y(t) \\
z(t)
\end{pmatrix} + \begin{pmatrix}
\kappa x(t) \\
\kappa y(t) \\
0
\end{pmatrix}
\]

2. Writing Equations (1) and (2), we may write:

\[
\begin{align*}
\dot{V_1}(t) &= \frac{d}{dt} \frac{1}{2} x^2(t) = \alpha x(t) x(t) + \kappa x(t) x(t) \\
\dot{V_2}(t) &= \frac{d}{dt} \frac{1}{2} y^2(t) = \beta y(t) y(t) + \kappa y(t) y(t) \\
\dot{V_3}(t) &= \frac{d}{dt} \frac{1}{2} z^2(t) = \gamma z(t) z(t) + \kappa z(t) z(t)
\end{align*}
\]

Since \( \frac{d}{dt} V(t) = \dot{V_1}(t) + \dot{V_2}(t) + \dot{V_3}(t) = 0 \), we have:

\[
0 = \alpha x(t) x(t) + \beta y(t) y(t) + \gamma z(t) z(t) + \kappa x(t) x(t) + \kappa y(t) y(t) + \kappa z(t) z(t)
\]

3. Similarly, given \( x(t) = \dot{x}(t) = 0 \), we have:

\[
\begin{align*}
\dot{x}(t) &= \alpha x(t) x(t) + \kappa x(t) x(t) \\
\dot{y}(t) &= \beta y(t) y(t) + \kappa y(t) y(t) \\
\dot{z}(t) &= \gamma z(t) z(t) + \kappa z(t) z(t)
\end{align*}
\]

4. The same holds in both tanks, is

\[
\begin{align*}
\dot{x_1}(t) &= \alpha x_1(t) x_1(t) + \kappa x_1(t) x_1(t) \\
\dot{y_1}(t) &= \beta y_1(t) y_1(t) + \kappa y_1(t) y_1(t) \\
\dot{z_1}(t) &= \gamma z_1(t) z_1(t) + \kappa z_1(t) z_1(t)
\end{align*}
\]

\[
\begin{align*}
\dot{x_2}(t) &= \alpha x_2(t) x_2(t) + \kappa x_2(t) x_2(t) \\
\dot{y_2}(t) &= \beta y_2(t) y_2(t) + \kappa y_2(t) y_2(t) \\
\dot{z_2}(t) &= \gamma z_2(t) z_2(t) + \kappa z_2(t) z_2(t)
\end{align*}
\]

\[
\begin{align*}
\dot{x_3}(t) &= \alpha x_3(t) x_3(t) + \kappa x_3(t) x_3(t) \\
\dot{y_3}(t) &= \beta y_3(t) y_3(t) + \kappa y_3(t) y_3(t) \\
\dot{z_3}(t) &= \gamma z_3(t) z_3(t) + \kappa z_3(t) z_3(t)
\end{align*}
\]
Previous Case (Equations 3 and 4)
These are the same as for the previous case.

\[ \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} = 0 \]

Similar to:

\[ \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} = 0 \]

\[ \frac{\partial y}{\partial t} = 0 \]

\[ \frac{\partial y}{\partial t} = \frac{\partial y}{\partial t} \]

Vol 1 = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial t} + \frac{\partial y}{\partial t} + \frac{\partial y}{\partial t}

Vol 2 = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial t}

The volumes are still:

\[ V_1(t) = \frac{\partial y}{\partial t} \]

\[ V_2(t) = \frac{\partial y}{\partial t} \]

And:

\[ \frac{\partial^2 y}{\partial t^2} = 0 \]

\[ \frac{\partial^2 y}{\partial t^2} = 0 \]

Obvious:

In tank 1 and tank 2, the density is:

A (uniform) and \( \rho \) (uniform) for both tanks.

Let's assume the problem is somewhat unrealistic. As such, let's assume the same seems somewhat unrealistically.

The assumption that the (uniform) density in both tanks is:

\[ \rho \]

\[ \rho \]

Note:
\[ \frac{dv}{dt} = m(t) + k \left[ h_2 - h_1 \right] \]

Substituting time arguments:

\[ \frac{dv}{dt} = m(t) - k [ g(t) - h_2(t)] \]

\[ g(t) = \int \frac{dv}{dt} dt \]

Since \[ \mu(\tau) \] is the unit step function:

\[ \mu(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases} \]

\[ \frac{dv}{dt} = m(t) - \rho(t) \left[ g(t) - h_2(t) \right] \]

The rate of change of dye in tank #1 depends on whether it is supplying dye to tank #2. If \( g(t) > 0 \), thus tank #1 is getting dye from tank #2. If \( g(t) < 0 \), thus tank #1 is losing dye.
The state equations for this time-invariant non-linear model, from (10), (11), and (12), are thus:

\[ \begin{align*}
\frac{dh(t)}{dt} &= \frac{v(t)}{h(t) - h_0(t)} - \frac{v(t)}{h(t) - h_0(t)} \\
\frac{dv(t)}{dt} &= k \left( g(t) - h(t) - h_0(t) \right) + \frac{v(t)}{(h(t) - h_0(t))^3} + \frac{v(t)}{h(t) - h_0(t)} \\
\frac{dh_0(t)}{dt} &= \frac{v(t)}{h(t) - h_0(t)} \\
\frac{d\theta(t)}{dt} &= \frac{v(t)}{h(t) - h_0(t)} \\
\end{align*} \]
\(0 = \frac{d}{dt} \left[ M - \frac{1}{2} (R_1 + R_1') + \frac{2}{3} R_2 \right] + \frac{\partial}{\partial v} \left[ \frac{1}{2} R_2 v^2 + \frac{1}{2} m v^2 \right] + \frac{\partial}{\partial v} \left[ \frac{1}{2} m v^2 \right]
\)

Substitute into

Solving for

\(0 = \frac{d}{dt} L_2 = \frac{\partial}{\partial v} \left[ \frac{1}{2} m v^2 \right] + \frac{\partial}{\partial v} \left[ \frac{1}{2} m v^2 \right]
\)

Variables

Use \(v, L_2\) and \(R_2\) as state variables.
Our state equations are:

\[ \frac{d^2 \theta}{dt^2} + \frac{M}{I_R} \frac{d \theta}{dt} + \frac{1}{I_R} \theta = \frac{E}{I_R} \]

\[ \frac{d^2 \phi}{dt^2} + \frac{M}{I_R} \frac{d \phi}{dt} + \frac{1}{I_R} \phi = \frac{E}{I_R} \]

Using: \( \frac{M}{I_R} = \frac{\dot{\phi}}{\dot{\theta}} \)

Substituting into:

\[ \frac{d^2 \theta}{dt^2} + \frac{M}{I_R} \frac{d \theta}{dt} + \frac{1}{I_R} \theta = \frac{E}{I_R} \]

Solving for \( \dot{\theta} \) and \( \dot{\phi} \):

\[ \ddot{\theta} = \frac{E}{I_R} - \frac{M}{I_R} \dot{\theta} - \frac{1}{I_R} \theta \]
In Laplace Form

\[ S\hat{x} = x + 5u \]

Signal Flow Graph:

\[ y(t) = x(t) \]

\[ y = \mathcal{L}\{y(t)\} = \mathcal{L}\{x(t)\} \]

\[ \mathcal{L}\{y(t)\} = \frac{1}{5} \]

\[ \mathcal{L}\{x(t)\} = \frac{s}{s^2 + 1} \]

\[ \frac{-s}{s^2 + 1} + \frac{5}{s} = s \]

\[ \mathcal{L}\{y(t)\} = \left( \frac{-s}{s^2 + 1} + \frac{5}{s} \right) \]

\[ \frac{s}{s^2 + 1} \]
BLOCK DIAGRAM IS THEN:

\[ (x+1) x^2 = 5u \Rightarrow x^2 = \frac{5}{1} \Rightarrow x = \pm \sqrt{5} \]

REARRANGING.
\[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2X \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} X \\ 1 \end{bmatrix} \]

**Laplace Transforming**

\[ Y = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2X \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} X \\ 1 \end{bmatrix} \]

Let

\[ \begin{bmatrix} M \\ 0 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ Y = 2 \begin{bmatrix} 2+5x \end{bmatrix} \]

\[ \begin{bmatrix} M \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ M = 2 + 5x \]

\[ \frac{2x}{y} = \frac{(2+5x)(1+5x)}{(y)(y)} \]

\[ \frac{(2+5x)}{(1+5x)} \]
To draw block diagram, we write:

\[ u(s) \]

\[ x_1 = x_2 = \frac{1}{s} x \]

\[ y = 2x_1 + 3x_2 + u \]
\[ \mathbf{Y}(s) = \frac{s^2 + 7s + 12}{s(s + 5)(s + 3)} = Y(s) \frac{1}{U(s)} \]

\[ \mathbf{Y}(s) = \begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \quad \mathbf{U}(s) = \begin{bmatrix} 0 & -2 & 3 \end{bmatrix} \]

\[ \mathbf{U}(s) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \]

\[ Y(s) = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \]

\[ \mathbf{x}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \]

OR, WITH \( \mathbf{x}_1 = \mathbf{w}_1 \)

\[ \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \]

\[ \mathbf{x}_3 = \mathbf{w}_3 \]

\[ \mathbf{X}(s) \text{ FORMING:} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \]
To draw block diagram, consider

\[
\begin{align*}
5x_3 &= -2x_2 - 3x_1 + u_1 \\
5x_2 &= x_3 \\
5x_1 &= x_2
\end{align*}
\]
Block Diagram

Signal Flow Graph

\[ \begin{align*}
  T &= y \\
  S &= y \\
  x &= y \\
  \frac{dx}{dt} &= x(t) + x(t) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
  \text{or} \\
  x(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) \\
  x &= c(t)
\]
CONTROLLABILITY TEST

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ AB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} B \mid AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(non-singular)} \]

\[ \Rightarrow \text{SYSTEM IS CONTROLLABLE} \]

OBSERVABILITY TEST

\[ Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow A^T C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ \begin{bmatrix} C^T \mid C^T A^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(non-singular)} \]

\[ \Rightarrow \text{SYSTEM IS OBSERVABLE} \]
HAS RANK 2

Equivalently, \( \det(A) \neq 0 \) or

Since this matrix is not controllable,

\[
\begin{bmatrix}
  A & B \\
  A_2 & A_1
\end{bmatrix} \Rightarrow
\begin{bmatrix}
  A_2 & \mathbf{0} \\
  \mathbf{0} & \mathbf{0}
\end{bmatrix}
\]

\[
A \mathbf{Z} \mathbf{B} =
\begin{bmatrix}
  A_2 \mathbf{Z} \mathbf{B}_1 \mathbf{0} \\
  \mathbf{0} & \mathbf{0}
\end{bmatrix}
\]

Test for controllability.

\[
\begin{bmatrix}
  x_1' \\
  x_2'
\end{bmatrix} = \begin{bmatrix}
  0 & -1 \\
  1 & 0
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\]

Therefore, Problem (c) for \( M = 0 \) is the answer to Problem (c).
The system is observable if
\[ \text{rank} = 4 \geq 2 \]
and the columns of \( A \) are linearly dependent.

Thus:
\[ \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ (A^T) C \]

\[ C' \]

Given
The block diagram is thus:

\[
\begin{align*}
\frac{z + 2}{z - 1} + \frac{z + 2}{z - 1} &= \frac{z + 2}{z - 1} \\
x_3 &= x_2 + x_1 \\
x_2 &= \frac{z + 2}{z - 1} \\
x_1 &= \frac{z + 2}{z - 1} \\
\end{align*}
\]

Now

Signal Flow Graph:

\[
\begin{align*}
x_3 &= -3x_1 + 4x_2 - 2x_3 + u_1 \\
x_2 &= -2x_1 + x_3 + u_2 \\
x_1 &= x_2(2) + u_3 \\
\end{align*}
\]

\[
\begin{bmatrix}
x(1) \\
x(2) \\
x(3)
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} 
\begin{bmatrix}
x(1) \\
x(2) \\
x(3)
\end{bmatrix}
\]
The Rank is 2

\[ \text{Not Controllable} \iff \text{Entire Second Row is Zero.} \]

\[
\begin{bmatrix}
1 & 0 & -2 & -3 & 12 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & -4 & 1
\end{bmatrix}
\]

\[
B'AB = AB
\]

\[
\begin{bmatrix}
12 \\
0 \\
0 \\
0 \\
-4
\end{bmatrix}
\]

\[
A
\]

\[
B =
\]

Testing for Controllability
The system is observable.

\[ \text{Rank} = 3 \]

\[
\begin{bmatrix}
  1 & 0 & 0 & -4 \\
  0 & 1 & 0 & -4 \\
  0 & 0 & 1 & 1 \\
-4 & 0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
  CT \\
  AT^2CT \\
  AT^2 \\
  1
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  1 \\
-2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 & 0 & -4 \\
  0 & 1 & 0 & -4 \\
  0 & 0 & 1 & 1 \\
  0 & -1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  -2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix} = \begin{bmatrix}
  1 & 0 & 0 & 0
\end{bmatrix} x(t)
\]

Testing for observability.
\[ V_{\text{MAX}} = H_{\text{MAX}} \]

A possible assignment is:
\[ 0 \leq V_2(t) \leq V_{\text{MAX}} \]
\[ 0 \leq V_1(t) \leq V_{\text{MAX}} \]

Die in the tanks must be:
\[ 0 \leq H_2(t) \leq H_{\text{MAX}} \]
\[ 0 \leq H_1(t) \leq H_{\text{MAX}} \]

Furthermore, since tanks 1 & 2 have finite heights:

The two states, \(h_1\) and \(h_2\),
\[ 0 \leq h_2(t) \leq W_{\text{MAX}} \]
\[ 0 \leq h_1(t) \leq W_{\text{MAX}} \]
\[ 0 \leq m(t) \leq M_{\text{MAX}} \]

Be negative. Thus upper bounded and cannot.

The input flows must be:
\[ \text{To minimize, } V \]

We would of course want.
\[ J = \int_{t_0}^{t_f + 24} V(t) \cdot dt \]

Period is as possible over a 24 hour.

As possible to measure To M.

2-4. (a) A performance measure To
\[
\begin{bmatrix}
Y_1 & 0 & 0 \\
0 & Y_2 & 0 \\
0 & 0 & Y_3
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
\begin{bmatrix}
\dot{Y}_1 \\
\dot{Y}_2 \\
\dot{Y}_3
\end{bmatrix} +
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\tau_1} & 0 & 0 \\
0 & \frac{1}{\tau_2} & 0 \\
0 & 0 & \frac{1}{\tau_3}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3
\end{bmatrix}
\]

From (1) and (2), with inputs \(v(t)\) and \(e(t)\), gives

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ I_3(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ I_3(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ I_3(t) \end{bmatrix}
\end{align*}
\]

Summing Torques:

- Opposite Cev. Torque
- Opposite Torque (opposes developed torque)
- Local Torque
- Opposes Developed Torque
- Developed Torque
- The Mechanical Torques Are

\[
\begin{align*}
\tau(t) &= \frac{d}{dt} \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{d^2}{dt^2} \varphi_1(t) \\ \frac{d^2}{dt^2} \varphi_2(t) \\ \frac{d^2}{dt^2} \varphi_3(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau_1} & 0 & 0 \\ 0 & \frac{1}{\tau_2} & 0 \\ 0 & 0 & \frac{1}{\tau_3} \end{bmatrix} \begin{bmatrix} I_1(t) \\ I_2(t) \\ I_3(t) \end{bmatrix}
\end{align*}
\]

(From the Circuit)
$\dot{q}_i = \int_0^t \left( k_w(t) - \dot{q}_i \right) dt$

would be a performance measure for this close as possible to $5 \text{ mph}$. The
we wish to keep the speed as

$\dot{v}(t) = k_w(t)$

A wheel's then

example, $k$ is the radius of
to linear velocity, $v(t)$, i.e., for
velocity is transformed linearly
first off, assume that the angular
invaildates the state equations
(This condition, by the way,

- $(c) L_i f_i = 0$

similarly, the angular velocity

$|f_i(t)| \leq I_{\text{max}}$

will be bounded:

state variable constraints

$|L_i(t)| \leq A_{\text{max}}$

likewise, $L_i(t)$ will have bounds. Let

As such, let the constraint be

$|e(t)| \leq E_{\text{max}}$

(b) input control constraint
Absorbed into \( A \). \( \therefore \frac{1}{\alpha} \) is the performance requirements. Between the two performances introduce to allow tradeoff where \( \mu \) is a weighting factor. 

\[
J = \int_0^\infty \left[ \mu x(t) + \left( \kappa x(t) - \frac{e^2}{2} \right) \right] dt \]

Energy is the sum of \( \mu \) and \( \kappa \). 

The composite performance measure is 

\[
J_2 = \int_0^\infty \frac{1}{\alpha} \int_0^\infty \kappa x(t) dt \]

Thus a second performance measure is 

\[
E = \int_0^\infty \int_0^\infty \kappa x(t) dt \]

Control during the mission is energy expended by the hour. For \( L = 0 \), the total hours of revolutions per has units of revolutions per \( t \). Has units of miles. And \( w(t) \) where \( t \) is the mission time.
Again, \( w \) is a weighting factor.

\[
J = \sqrt{\int_{t_1}^{t_2} \left[ \frac{m(t)}{v(t)^2} \right] dt}
\]

Sum of this and (3):

The total performance.

\[
L = \int_{t_1}^{t_2} E(t) dt
\]

General form.

Energy, however, must now be written in the more valid, the total expended.

For \( L > 0 \), as still.
\[ \dot{x} = \frac{v(t)}{x(t)} \]

Performance Measure \( \mathcal{C}_E \)

To \( u(t) \), let the energy be here proportional to \( u(t)^2 \).

And since the control energy is here proportional under the input constraint, to attain the state constraint (b) in that we require \( \max \theta \)

\[ \theta = \dot{\theta} \]

On \( x(t) = \theta(t) \) will not place any constraints on \( \dot{x}(t) \).

Velocity requirement, we since there is no angular \( 1 \leq 90, 2 \leq 30, 5 \leq 15 \)

At \( t = 4 \). Thus the state \( x(t) = \Theta(t) \) is specified.

\[ |u(t)| = \theta \text{ max} \]

Would be bounded, so the input constraint gas less must obviously be

\[ (2-4)(2) \] The torque produced at the
For minimum time would be

\( u(t) = \frac{J}{t_f - t_0} \)

OR, setting \( t_0 = 0 \):

\( u(t) = \frac{J}{t_f - t_0} \)

(b) The best performance measure

\[ 10 \leq \Theta(t_f) \leq 15 \text{.}^\circ \]

The problem:

Here are the same as in

(2.5) (a) The input and state constraints

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At \( t = 0 \), \( x' = 0 \), \( y' = 0 \):

Also, since the rocket starts

\[ O < M_{\text{in}} \leq x \leq M_{\text{max}} \]

Plus full fuel load:

By the rocket's mass

Mass and upper bounded by the rocket's

The mass, \( m(t) \), is lower

velocity, \( x' \)

As is the horizontal

Similarly unbounded

\( x \neq \text{horizontal distance is} \)

is unbounded

In principle, the velocity, \( x' \)

\[ x' = 20 \]

We assume \( x = 0 \) is the ground:

STATE CONSTRAINTS

\( u \) and \( \omega \) always appear as the argument of a trig. function

Actually, this doesn't matter since

\[ 0 < u < 2 \pi \]

The thrust angle must be

\[ 0 < \theta \leq 90^\circ \]

Upper bounded

The thrust must be positive and

(2-6) (3) inert constraints

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Again, we wish to minimize \( J \):

\[
J = \int x^5 (t_f) dt
\]

To maximize \( M(t) = x \) is

A performance measure

where \( t_f = 2.5 \text{ min} \)

\[
x(f) = 3 \text{ miles} < x^1(t_f)
\]

\[
x(f) = 500 \text{ miles} = x^1(t_f)
\]

(c) Additional constraints are

Minimize \( J \).

To denote that we wish to

where we use the minus sign

\[
J = -x(f) = -x^1(t_f)
\]

A performance measure is

if \( f \) is specified, our

\[
y(f) = x^3(t_f) = 3 \text{ miles}
\]

Obviously

(b) An additional constraint is

\[
x_2(t_0) = 0 \quad x_4(t_0) = 0
\]

Since it starts from rest:
and that the resulting chain. Any action is accidental.

with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $s = 0$, $a = I$, $r = 1$, $t = \infty$

Determine the optimal control. Here a new action

$\delta$ indicates to the optimum

$p = -\nabla A - \nabla B + \nabla P - \nabla C$

by adjoining a matrix. A useful

show that one may note the optimal feedback control having

When $s$ and $a$ are given, the adjoint function $\gamma(t) =

\frac{1}{2} \int_0^T \left( \frac{\partial^2 x}{\partial t^2} + \frac{\partial y}{\partial x} \right) dt$

and the performance measure to be minimized is

$J = \frac{1}{2} x + \frac{1}{2} u$

Accordingly by

consider the function $u(x)$ in which the point is

$a(x) = 0$,

which have only one canonical. The Hamiltonian conditions

$\frac{\partial}{\partial x} \left( \int_0^T (x^2 + (x')^2) dt \right) = 0$

because the adjoint $\frac{\partial}{\partial x} (x + u) = 0$

and the adjoint

$\phi (t) = 0$

satisfy the Hamiltonian equations $(x', y', \phi) = 0$. For $u(x) = 0$

the Hamiltonian extended

$\phi (t) = 0$, $x = 0$, $y = 0$.

\( \text{(a)} \) Find on $\gamma (x, t)$ which yield the Euler-Lagrange equation

\( \text{(b)} \) Find another
We will choose $a$ to make

$$\text{Let } a = \frac{1}{2} \text{ and } \gamma = \frac{3}{2}$$

If $a = 0$, then $x(t) = 0$

$$0 = x(t) = a \ln(t)$$

$$x = \frac{1}{2} x$$

$$x(0) = 0 \iff x = 0$$

$$x = a \ln(t) + b$$

$$x = -ax + x + x + 0 = 0$$

$$\int_{-x}^{x} f(x) \, dx = \int_{-x}^{x} f(x) \, dx$$

$$\int_{-x}^{x} f(x) \, dx = \int_{-x}^{x} f(x) \, dx$$

$$f(x) = \frac{1}{2} f(x) + \phi \left( x \right)$$

$$\phi(0) = 0$$
Change into average grade.

The average grade \( \Phi \) is

\[
\Phi(x) = 0
\]

\( x = 0 \), \( x = 0 \)

The average grade function, \( \Phi(x) \), is defined as

\[
\Phi(x) = \begin{cases}
  0 & x < 0 \\
  \frac{x}{x^2 + \left( \frac{1}{x} \right)^2} & 0 \leq x
\end{cases}
\]
Given: By applying van der Waerden's Theorem

\[ x = (1, -1) \] as what we did.

Two solutions:

\[ x(4) = 2 \]
\[ x(0) = 0 \]

For \( x = 1 - 2 \) (Note, in general, \( x = 0 \))

Just plug in:

\[ \frac{x}{y} = \frac{2}{0} \]
\[ x = \frac{2}{0} \]

Most obey such. Prove: \( \frac{x}{y} = 0 \)

This has \( x = 0 \)
\[ 0 = \phi^3 + 3\phi + 1 \]

\[ \phi = \frac{-3 \pm \sqrt{9 - 4 \cdot 1 \cdot 1}}{2} \]

\[ \phi = \frac{-3 \pm \sqrt{5}}{2} \]

\[ \phi = \frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2} \]

The golden ratio is

\[ \phi = \frac{3 + \sqrt{5}}{2} \]

The other solution is

\[ \phi = \frac{3 - \sqrt{5}}{2} \]

The curve conditions:

\[ x = \frac{3 \pm \sqrt{5}}{2} \]

**Diagram:**

- Points labeled 0, 1, 2, 3, 4, 5, 6, 7.
- Lines connecting the points.
- The golden ratio and curve conditions are plotted on the diagram.
\( p(x) = 5 \)

\[ \left\{ \begin{array}{l}
\frac{1}{x + a + 4T - \text{pe}} + a + \frac{1}{x} + Q = 0 \\
\end{array} \right. \]

**Theorem:**

If \( p(x) = 5 \), then \( x = \frac{5}{x} \).

\[ \begin{align*}
\sqrt{\frac{1}{x + a + 4T - \text{pe}} + a + \frac{1}{x}} + \sqrt{\frac{1}{x + a + 4T - \text{pe}}} + a + \frac{1}{x} &= 0 \\
A &= a + x \\
\text{Assume} \quad x = a + e
\end{align*} \]

\[ y = \frac{\frac{1}{x + 4T} + a + \frac{1}{x}}{\frac{\frac{1}{x + 4T} + a + \frac{1}{x}}{x + 4T}} \quad u = -a - e \\
\]

\[ z = H = \frac{x + \sqrt{T + i \sigma + x}}{x + \sqrt{T + i \sigma + x}} \]
\[ Y = AX + B \]

\[ Y_1 = -X'_{1v} + Y_{1v} + Y'_{1v} \]

\[ Y_2 = X_{2v} \]

\[ X = [0 0] + X' \]

\[ Y = [1 1] + Y' \]

\[ \sqrt{2} Y = [0, 1] \]

\[ v = P - R \]

\[ y = \begin{pmatrix} 1 & 1 \end{pmatrix} \]

Where \( X \) and \( Y \) are matrices.
The Schrödinger equation is used to solve the problem and the wave function 

\( \Psi(x, t) \) is found. The wave function \( \Psi(x, t) \) can be normalized. The potential \( V(x) \) can be considered by Hamiltonian which is \[ H = -\frac{\hbar^2}{2m} \nabla^2 + V(x). \]

\[ \Psi(x, t) = c. \]