# Holographic representation of space-variant systems: system theory

Robert J. Marks II and Thomas F. Krile

**System theory for holographic representation of linear space-variant systems is derived. The utility of the resulting piecewise isoplanatic approximation (PIA) is illustrated by example application to the invariant system, ideal magnifier, and Fourier transformer. A method previously employed to holographically represent a space-variant system, the discrete approximation, is shown to be a special case of the PIA.**

## I. Introduction

By holographically recording an arbitrarily complex linear space-invariant system's transfer function, one may duplicate input-output relationships of the system with access solely to the hologram. An arbitrary input is Fourier transformed by a thin lens, multiplied by the holographically recorded transfer function placed in the lens's back focal plane, and reimaged. The output is equivalent to that of the recorded system with the same input. This type of system representation may be used to condense a multielement space-invariant linear system into a single hologram.

Unfortunately, there are many common optical systems of interest that are space variant to which this scheme cannot be directly applied. Recent efforts to holographically record space-variant systems employing a variation of the transfer function method, however, have proven successful. Either by input isoplanatic patch division or sampling, successful volume hologram' representations of the space-variant nonunity magnification imaging system have been obtained.[1-5]

It is thus of practical importance to investigate that system theory necessary for holographically representing space-variant systems. The approach employed herein divides the space-variant system input plane into isoplanatic patches to which corresponding transfer functions are assigned. The resulting piecewise isoplanatic approximation (PIA) output is representative of the true system output. Although the isoplanatic patch concept is not new,[6] its use has been primarily confined to system analysis within a single patch.

The PIA is herein derived, and examples of its application to various systems are offered. Equivalence of true and PIA outputs of space-invariant systems is established. The discrete approximation (DA) previously employed for space-variant system representation[4] is shown to be a special case of the PIA.

## II. Foundations

For notational and continuity purposes, a brief review of classical linear system theory is now offered after Goodman.[7] Analysis is restricted to one dimension with no loss of generality.

A general system consisting of an input, a black box, and an output may be modeled by the mathematical operator S [ ] such that

$$g_0(x) = \mathbf{S}[g_i(x)], \tag{1}$$

where $g_0(x)$ is an output corresponding to an input $g_i(x)$. A system is said to be linear if and only if

$$\mathbf{S}[ag_i(x) + bf_i(x)] = a\mathbf{S}[g_i(x)] + b\mathbf{S}[f_i(x)], \tag{2}$$

where $a$ and $b$ are constant. Such systems may be described by the superposition integral

$$g_0(x) = \int_{-\infty}^{\infty} g_i(\xi) h(x-\xi;\xi) d\xi, \tag{3}$$

where, after the notation of Lohmann and Paris,[8]

$$h(x-\xi;\xi) = \mathbf{S}[\delta(x-\xi)] \tag{4}$$

is the system's line spread function, and $\delta(x)$ is the Dirac delta.

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When this work was done, both authors were with the Rose-Hulman Institute of Technology, Department of Electrical Engineering, Terre Haute, Indiana 47803. R. J. Marks II is presently with Texas Tech University, Department of Electrical Engineering, Lubbock, Texas 79409.

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A linear system is said to be space invariant, or isoplanatic, if

$$h(x-\xi;\xi) = h(x-\xi), \tag{5}$$

that is, the line spread function is a function only of the difference $(x-\xi)$. For such systems, Eq. (3) becomes the convolution integral

$$g_0(x) = \int_{-\infty}^{\infty} g_i(\xi)h(x-\xi)d\xi, \tag{6}$$

which is expressed in shorthand notation as

$$g_0(x) = g_i(x) * h(x). \tag{7}$$

For the space-variant linear system, one needs to know the line spread function from every point on the input plane to define completely the system, while the invariant case demands only one defining relationship.

The input-output expression of Eq. (7) may also be expressed in the frequency domain as

$$G_0(f_x) = H(f_x)G_i(f_x), \tag{8}$$

where $G_0(f_x)$, $H(f_x)$, and $G_i(f_x)$ are the respective Fourier transforms of $g_0(x)$, h(x), and $g_i(x)$. For example,

$$H(f_x) = \int_{-\infty}^{\infty} h(x)\exp(-j2\pi f_x x)dx \tag{9}$$

is termed the system transfer function.

The straightforward relationships in Eqs. (7) and (8) give rise to the feasibility of holographic recording of space-invariant systems described in Sec. I. A further generalization is needed for recording techniques for the space-variant case.

## III. Piecewise Isoplanatic Approximation

In the above analysis, a system was given a specific classification of variant or invariant by the criterion of Eq. (5). Lohmann and Paris[8] have proposed that variant systems might be assigned degrees of invariance. It follows that a space-variant linear system having a high degree of invariance might successfully be analyzed in a manner similar to that used for invariant systems. To do this, the input plane is divided into a number of regions, or isoplanatic patches, in which the line spread function essentially meets the invariance criterion. Each input function is divided into similar regions that are convolved with corresponding line spread functions and superimposed to yield an approximated output.

### A. Derivation

A linear system composed of invariant input regions is termed piecewise isoplanatic. The distribution of these regions on the input plane may be expressed as

$$\sum_n \mu(x-l_n)\mu(-x+u_n), \tag{10}$$

where 1, and $u_n$ are, respectively, the lower and upper endpoints of the nth patch, and $\mu(x)$ is the unit step function defined as

$$\mu(x) = \begin{cases} 1; x \geq 0, \\ 0; x < 0. \end{cases} \tag{11}$$

The summation is assumed to cover the region of interest on the input plane. We also assume adjacent patches are joined so that

$$u_n = l_{n+1} \tag{12}$$

The nth isoplanatic patch thus has a width of $u_n - l_n$ and is centered at $x$ = (ln+ $u_n$)/2.

In order to define completely the piecewise isoplanatic system, knowledge of the system's line-spread function for each patch is needed. We thus assume line sources are conveniently placed at some input plane point $x_n$ where

$$l_n \leq x_n \leq u_n, \tag{13}$$

and that we have knowledge of

$$\begin{aligned} h_n(x-x_n) &= h(x-x_n;x_n) \\ &= \mathbf{S}[\delta(x-x_n)] \end{aligned} \tag{14}$$

for all $n$.

Consider now the input-output relationship of the piecewise isoplanatic system. An input, $g_i(x)$, must first be divided into isoplanatic regions. Specifically,

$$g_i(x) = \sum_n g_n(x-x_n), \tag{15}$$

where

$$g_n(x-x_n) = g_i(x)\mu(x-l_n)\mu(-x+u_n). \tag{16}$$

An illustration of extraction of g,(r) from $g_i(x)$ is offered in Fig. 1.

The output of the piecewise invariant system is found through substitution of Eq. (15) into Eq. (1);

$$g_0(x) = \sum_n \mathbf{S}[g_n(x-x_n)]. \tag{17}$$

The summation sign is extracted from the system operator due to superposition [(Eq. (2)]. Each argument of the system operator is invariant and can be expressed via the convolution integral [Eq. (6)]:

$$\mathbf{S}[g_n(x-x_n)] = \int_{-\infty}^{\infty} g_n(\xi-x_n)h_n(x-\xi)d\xi. \tag{18}$$

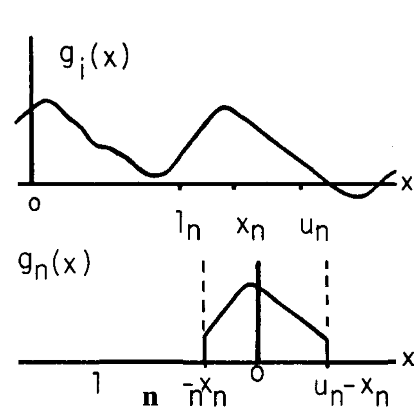


Fig. 1. Extraction of the nth isoplanatic region g,,(x) from an input $g_i(x)$.

Substituting into Eq. (17) and employing the shorthand notation of Eq. (7) give

$$g_0(x) = \sum_n g_n(x - x_n) * h_n(x) \tag{19a}$$

or equivalently

$$g_0(x) = \sum_n g_n(x) * h_n(x - x_n). \tag{19b}$$

The input-output relationship of a piecewise isoplanatic system can thus be expressed as a superposition of convolutions.

Equations (19a) and (19b) may also be expressed in the frequency domain as

$$G_0(f_x) = \sum_n G_n(f_x) H_n(f_x) \exp(-j2\pi f_x x_n), \tag{20}$$

where $G_n(f_x)$ and $H_n(f_x)$ are the respective Fourier transforms of $g_n$ (x) and $h_n(x)$.

The power of the input-output relationship of a piecewise isoplanatic system lies in its use for synthesizing outputs of variant systems modeled as piecewise isoplanatic. Such a system model output will be called the Piecewise Isoplanatic Approximation (PIA) Note that the superposition integral [Eq. (3)] can be written

$$g_0(x) = \sum_n \int_{l_n}^{u_n} g_i(\xi) h(x - \xi; \xi) d\xi. \tag{21}$$

The PIA of the same system, $\tilde{g}_0(x)$, can be written

$$\tilde{g}_0(x) = \sum_n \int_{l_n}^{u_n} g_i(\xi) h(x - \xi; x_n) d\xi. \tag{22}$$

One sees that as each isoplanatic patch width narrows around $x_n$ (matched by an increase in the number of patches), the PIA [Eq. (22)] approaches the true output $g_0(x)$.

## B. Examples

### 1. *Invariant Systems*

Application of the PIA to invariant systems gives the true output since invariant systems are indeed piecewise isoplanatic. For such systems, the invariance criterion of Eq. (5) is true. Substituting into the PIA [Eq. (19a)] and recognizing the distributive property of the convolution operator give

$$\begin{aligned} \tilde{g}_0 &= h(x) * \left[\sum_n g_n(x - x_n)\right] \\ &= h(x) * g_i(x) \\ &= g_0(x). \end{aligned} \tag{23}$$

The true and approximated outputs are thus equivalent.

Consider as a specific example the differentiator with an input-output relationship of

$$g_0(x) = (d/dx) g_i(x). \tag{24}$$

The corresponding line-spread function is

$$h(x - \xi) = \delta'(x - \xi), \tag{25}$$

where $\delta'(x)$ is the unit doublet, the first derivative of the Dirac delta. By the criterion of Eq. (5), the differentiator is invariant.

Substitution of Eq. (25) into the PIA [Eq. (19b)] yields

$$\begin{aligned} \tilde{g}_0 &= \sum_n g_i(x)\mu(x - l_n)\mu(u_n - x) * \delta'(x) \\ &= \sum_n \frac{d}{dx} g_i(x)[\mu(x - l_n) - \mu(x - u_n)] \\ &= \sum_n \mu(x - l_n)\mu(u_n - x)\frac{d}{dx} g_i(x) \\ &\quad + g_i(l_n)\delta(x - l_n) - g_i(u_n)\delta(x - u_n). \end{aligned} \tag{26}$$

In the above summation, the unwanted delta terms are canceled by corresponding delta terms in adjacent patches as specified in Eq. (12). This reduces Eq. (26) to

$$\begin{aligned} g_0(x) &= \sum_n \mu(x - l_n)\mu(u_n - x)\frac{d}{dx} g_i(x) \\ &= \frac{d}{dx} g_i(x), \end{aligned} \tag{27}$$

which is the desired result.

### 2. *Ideal Magnifier*

The ideal magnifier (imaging system) has an input-output relationship of

$$g_0(x) = (1/M) g_i(x/M). \tag{28}$$

The corresponding line-spread function is then

$$h(x - \xi; \xi) = \delta(x - M\xi). \tag{29}$$

Note that no mathematical manipulation may be performed on Eq. (29) to meet the invariance criterion of Eq. (5) except for the trivial case of unity magnification. Thus, in the classical sense, the magnifier is space variant.9

The PIA, $\tilde{g}_0(x)$, of the magnifier may be found by substituting the appropriate form of Eq. (29) into Eq. (19b):

$$\tilde{g}_0(x) = \sum_n g_n(x) * \delta(x - Mx_n). \tag{30}$$

Substituting Eq. (16) and evaluating the resulting convolution integral [Eq. (6)] give

$$\begin{aligned} \tilde{g}_0(x) = \sum_n g_i[x - (M - 1)x_n]\mu[x - l_n \\ - (M - 1)x_n]\mu[-x + u_n + (M - 1)x_n]. \end{aligned} \tag{31}$$

An illustration of the true and PIA outputs of the magnifier for a typical input is offered in Fig. 2 for M > 1. Each isoplanatically modeled region is "magnified" by being shifted a factor of *M*.

### 3. *Fourier Transformer*

The thin Fourier transforming lens may be viewed not only as a tool for optical information processing, but also as a space-variant linear system to which the PIA might be applied. The input-output relationship of a thin lens Fourier transformer may be expressed as

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