

Closed-form object restoration from limited spatial and spectral information

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Given only a portion of an arbitrary finite-energy $[L_2(-\infty, \infty)]$ image and a portion of that image's spectrum, we present a closed-form method by which the entire image can be reconstructed. The algorithm is a basic augmentation of a recently proposed iterative restoration algorithm presented by Stark *et al.* [J. Opt. Soc. Am. 71, 635 (1981); Opt. Lett. 6, 259 (1981)]. Experimental results are presented.

Building on the mathematical foundation laid by Youla,¹ Stark *et al.*^{2,3} recently presented an iterative algorithm for reconstruction of an L_2 signal with only partial knowledge of the signal in the spatial and frequency domain. In this Letter we show that the algorithms can be straightforwardly placed in closed form when they are implemented digitally. This treatment of the algorithm of Stark *et al.* is directly parallel to Sabri and Steenaart's⁴ closed-form treatment of Gerchberg's iterative extrapolation algorithm.^{5,6}

Development

A closed-form restoration matrix will first be developed for example A of Ref. 2. Following the identical notation, we have, from Eq. (24) of Ref. 2,

$$f_k(x) = \sum_{r=0}^{k-1} (Q_b P_a)^r m_1(x),$$

where m_1 is formed from the two known projections of $f(x)$:

$$m_1(x) = P_b f(x) + Q_b [Q_a f(x)]. \quad (1)$$

When they are implemented discretely in one dimension, the projection operators take on the form of matrices. The matrix P_a corresponding to the operator \mathcal{P}_a is simply⁴

$$P_a = \text{diag}\{0,0, \dots, 0,1, \dots, 1,1,1, \dots, 1,0, \dots, 0,0\},$$

i.e., a matrix everywhere zero except for 1's appropriately placed along the diagonal. The operator

$$P_b = 1 - Q_b$$

is simply a low-pass filter matrix that can be formed by

$$P_b = D^{-1} B D,$$

where D is the discrete Fourier-transform matrix and

$$B = \text{diag}\{0,0, \dots, 0,1, \dots, 1,1,1, \dots, 1,0, \dots, 0,0\}$$

deletes all frequency components outside the interval

$|\omega| \leq b$. Other formulations of the low-pass matrix are also possible.⁷ The Q_b matrix is obtained by

$$Q_b = I - P_b,$$

where I is the identity matrix.

Restoration Matrix

Let \mathbf{f}_k denote a vector of samples from $f_k(x)$. The restoration algorithm then becomes

$$\begin{aligned} \mathbf{f}_k &= \sum_{r=0}^{k-1} (Q_b P_a)^r \mathbf{m}_1 \\ &= R_k \mathbf{m}_1, \end{aligned}$$

where the k th-order restoration matrix

$$R_k = \sum_{r=0}^{k-1} (Q_b P_a)^r$$

can be computed off line.

If we let k tend to ∞ , then, assuming convergence,⁴

$$\begin{aligned} R &\triangleq R_\infty \\ &= \sum_{r=0}^{\infty} (Q_b P_a)^r \\ &= (I - Q_b P_a)^{-1}, \end{aligned}$$

where we have utilized a generalized geometric series. Then

$$\mathbf{f}_\infty = R \mathbf{m}_1. \quad (2)$$

The restoration matrix can thus be formed iteratively or, as here, through matrix inversion.

The above analysis can be generalized such that $f(x)$ and $F(\omega)$ are known over any interval. The choice of these intervals, however, dictates whether the problem is well or ill posed.^{1,2}

Experimental Results

To illustrate application of Eq. (2), we now present restoration of a number of degraded signals. Each

consists of 34 samples. The duration corresponding to \mathcal{P}_a in each example is 8 of the 34 points. Filtering was performed with a Sabri-Steenaart low-pass filter⁷ with bandwidth $\beta = 7$. In each of the six figures, points are connected for clarity of presentation.

The topmost function in Figs. 1-3 is the original image. Figure 1 is a single pulse, Fig. 2 is a single-sided

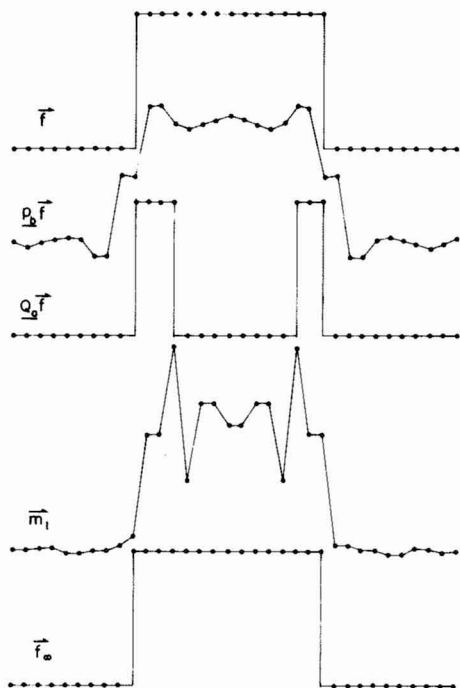


Fig. 1. Restoration of a pulse from $\hat{P}_b \mathbf{f}$ and $Q_a \mathbf{f}$.

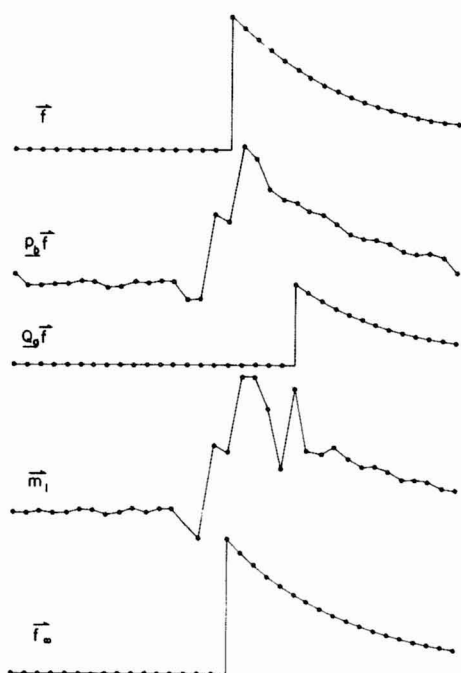


Fig. 2. Restoration of a single-sided exponential from $P_b \mathbf{f}$ and $Q_a \mathbf{f}$.

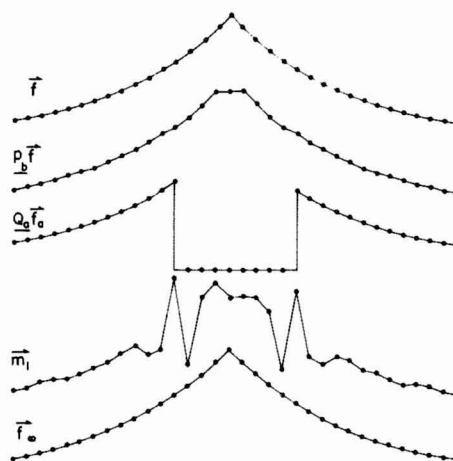


Fig. 3. Restoration of a double-sided exponential from $P_b \mathbf{f}$ and $Q_a \mathbf{f}$.

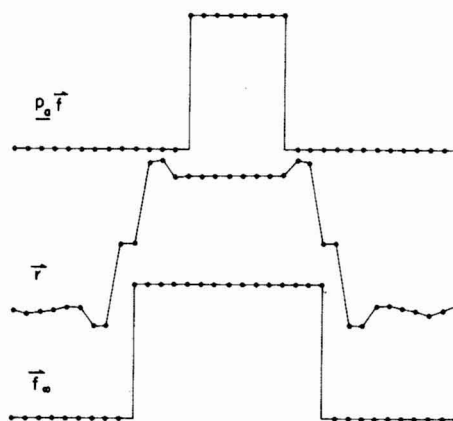


Fig. 4. Restoration of a pulse from $P_b \mathbf{f}$ and $P_a \mathbf{f}$. The vectors \mathbf{f} and $P_b \mathbf{f}$ are shown in Fig. 1.

exponential, and Fig. 3 is a double-sided exponential. Each figure also contains the assumed known portions of \mathbf{f} : $P_b \mathbf{f}$ and $Q_a \mathbf{f}$. The function m_1 is the linear combination of these projections corresponding to¹

$$\mathbf{m}_1 = P_b \mathbf{f} + Q_b Q_a \mathbf{f}.$$

The bottom function in Figs. 1-3 shows the restored image computed by Eq. (2). In each case, the restoration agreed with the image to six places.

A second set of examples was performed for the ill-posed problem presented in Ref. 3 reconstructing f from $\mathcal{P}_a f$ and $\mathcal{P}_b f$. When an identical procedure is followed, as before, the restoration matrix here becomes

$$R = (I - Q_a Q_b)^{-1}.$$

We form a linear combination of the known data, as in Eq. (10) of Ref. 3:

$$\mathbf{r} = P_a \mathbf{f} + Q_a P_b \mathbf{f}$$

and obtain the restoration

