Ambiguity function display: an improved coherent processor

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A coherent optical processor for displaying a signal's ambiguity function is described. The required time delay is realized by 45° rotations of two identical input transparencies and the Doppler shift by a 1-D Fourier transformation. The entire ambiguity function is displayed in the output (Doppler shift-time delay) plane. Examples of the optically computed ambiguity function for single and double pulse signals are shown to be in excellent agreement with theory. Advantages of this approach over other schemes and possible extension to real time processing are also discussed.

I. Introduction

The ambiguity function, first introduced by Woodward,[1] has been applied in radar in predicting the capability of a given signal to determine simultaneously the range and velocity of a target. The range is determined by the time delay τ and the velocity by the Doppler shift ν. The ambiguity function for a given real-valued signal $f(t)$ is

$$\chi(\nu,\tau) = \int_{-\infty}^{\infty} f(t)f(t-\tau)\exp(-j2\pi\nu t)dt, \tag{1}$$

In optics, Papoulis has employed the ambiguity function in analyzing diffraction phenomena.[2]

In this paper, we describe a rather easily implemented coherent processor capable of generating the ambiguity function in magnitude. A similar, yet somewhat more elaborate, scheme for generating $\chi(\nu,\tau)$ in both magnitude and phase is given in the Appendix. Such a scheme, for example, would need to be utilized when further coherent processing of the ambiguity function is required.

Cutrona *et al.*[3,4] and Preston[5] have proposed a coherent ambiguity function processor[6] which utilizes multiple channels to display the ambiguity function for discrete values of τ. The scheme of Casasent *et al.*[7] generates 1-D slices of the ambiguity function in the (ν,τ) plane. Similar 1-D displays have also been electronically produced.[8] Our method, as described in the following sections, (1) displays $|\chi(\nu,\tau)|^2$ in a continuous (rather than quantized) form over the entire (ν,τ) plane, (2) has the capacity for extension to real time processing, and (3) is easily implemented.

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II. Geometrical Interpretation

On the (t,τ) plane, a function $f(t)$ takes on the 1-D nature exemplified in Fig. 1(a). Upon rotating this function counterclockwise about the origin through an angle θ, we generate the function [Fig. 1(b)]

$$f(t\cos\theta + \tau\sin\theta). \tag{2}$$

Thus, for a rotation of 45°, we obtain $f[(t+\tau)/\sqrt{2}]$, and for a rotation of −45°, we obtain $f[(t-\tau)/\sqrt{2}]$. Consider, then, multiplying these two functions [Fig. 1(c)] and performing a Fourier transformation with respect to t:

$$\int_{-\infty}^{\infty} f\left(\frac{t+\tau}{\sqrt{2}}\right) f\left(\frac{t-\tau}{\sqrt{2}}\right)\exp(-j2\pi\nu t)dt, \tag{3}$$

where ν is the frequency variable associated with t. Upon making the variable substitution $t' = (t+\tau)/\sqrt{2}$, Eq. (3) becomes a scaled version of the ambiguity function of Eq. (1):

$$\sqrt{2}\exp(j2\pi\nu\tau)\int_{-\infty}^{\infty} f(t')f(t'-\sqrt{2}\tau)\exp[-j2\pi(\sqrt{2}\nu)t']dt'$$
$$= \sqrt{2}\chi(\sqrt{2}\nu,\sqrt{2}\tau)\exp(j2\pi\nu\tau). \tag{4}$$

Thus, apart from a multiplicative phase term, we may generate a scaled version of the ambiguity function by representing the time delay by simple 45° rotations and the Doppler shift by an appropriate 1-D Fourier transformation.

III. Implementation Scheme

A processor capable of performing a 1-D Fourier transform is given in Fig. 2. The field amplitude $U(\nu,\tau)$ in plane P_2 is related to the coherently illuminated transmittance $s(t,\tau)$ in plane P_1 by

$$U(\nu,\tau) = \exp(-j2\pi\lambda f\nu^2)\int_{-\infty}^{\infty} s(t,-\tau)\exp(-j2\pi\nu t)dt, \tag{5}$$

where λ is the wavelength of the spatially coherent il-

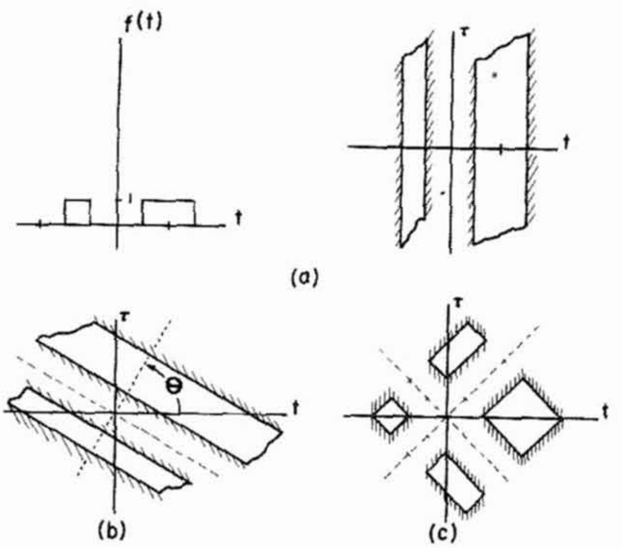

Fig. 1. (a) A function $f(t)$ in time and in the (t,τ) plane. (b) By rotating $f(t)$ counterclockwise an angle of θ about the origin of the (t,τ) plane, we generate $f(t\,\cos\theta + \tau\,\sin\theta)$. (c) The function $f[(t+\tau)/\sqrt{2}]f[(t-\tau)/\sqrt{2}]$ in the (t,τ) plane.

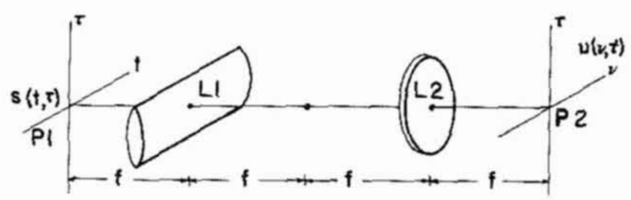

Fig. 2. A coherent processor for ambiguity function display. Both the lenses have focal length f. Fourier transformation is performed in the horizontal direction and imaging in the vertical direction.

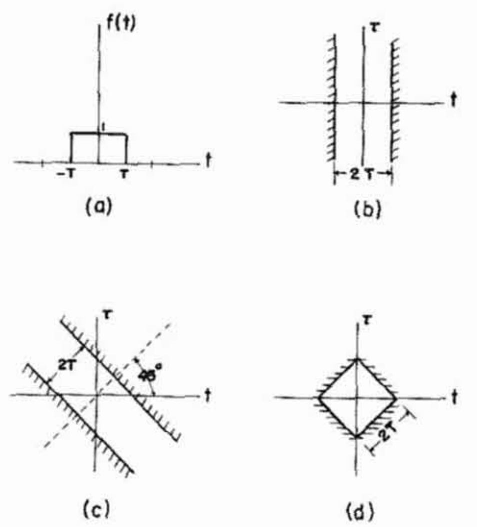

Fig. 3. A single pulse (a) in time, (b) in the (t,τ) plane, (c) rotated 45° on the (t,τ) plane to form $f[(t+\tau)/\sqrt{2}]$, (d) the product of two pulses rotated 45° and −45° on the (t,τ) plane to form $f[(t+\tau)/\sqrt{2}]f[(t-\tau)/\sqrt{2}]$.

lumination, f is the focal length of both lenses L_1 and L_2, and the spatial frequency ν is related to the horizontal displacement x_2 on plane P_2 by[9]

$$\nu = x_2/\lambda f. \tag{6}$$

Consider, then, placing two identical 1-D transparencies of $f(t)$ in plane P_1, each rotated 45° in such a manner as to form the product

$$s(t,\tau) = f\left(\frac{t+\tau}{\sqrt{2}}\right) f\left(\frac{t-\tau}{\sqrt{2}}\right). \tag{7}$$

The corresponding field amplitude in plane P_2 [from Eq. (5)] is then given by

$$\begin{aligned} U(\nu,\tau) &= \\ &\exp(-j2\pi\lambda f\nu^2)\int_{-\infty}^{\infty} f\left(\frac{t-\tau}{\sqrt{2}}\right) f\left(\frac{t+\tau}{\sqrt{2}}\right) \times \exp(-j2\pi\nu t)dt \\ &= \sqrt{2}\exp[-j2\pi\nu(\tau+\lambda f\nu)] \\ &\times \int_{-\infty}^{\infty} f(t')f(t'-\sqrt{2}\tau)\exp[-j2\pi(\sqrt{2}\nu)t']dt', \end{aligned} \tag{8}$$

where, as before, we have made the change of variable $t' = (t+\tau)/\sqrt{2}$. The intensity distribution associated with Eq. (8) is immediately recognized as a scaled version of the squared modulus of the ambiguity function[10]:

$$\begin{aligned} I(\nu,\tau) &= |U(\nu,\tau)|^2 \\ &= 2|\chi(\sqrt{2}\nu,\sqrt{2}\tau)|^2. \end{aligned} \tag{9}$$

IV. Experimental Results

To evaluate the performance of the proposed processor, the ambiguity functions for a single and double pulse signal are evaluated analytically and compared to the corresponding optical system outputs. In practice, the processor output is magnified by conventional means for observation and photographic purposes.

A. Single Pulse

For a single pulse [Fig. 3(a)], we may write

$$f(t) = \mathrm{rect}(t/2T), \tag{10}$$

where $2T$ is the pulse duration, and

$$\mathrm{rect}(t) \triangleq \begin{cases} 1; & |t| \le 1/2, \\ 0; & |t| > 1/2. \end{cases}$$

The geometric interpretations of $f(t)$, $f[(t+\tau)/\sqrt{2}]$, and $f[(t+\tau)/\sqrt{2}]f[(t-\tau)/\sqrt{2}]$ are shown in Figs. 3(b), 3(c), and 3(d), respectively.

Substituting Eq. (10) into Eq. (1) followed by evaluation yields the ambiguity function

$$\chi(\nu,\tau) = \begin{cases} (2T-|\tau|)\,\mathrm{sinc}\,\nu(2T-|\tau|)\exp(-j\pi\nu\tau); & |\tau| \le 2T \\ 0; & |\tau| \ge 2T, \end{cases} \tag{11}$$

where

$$\mathrm{sinc}\,\nu \triangleq (\sin\pi\nu)/\pi\nu.$$

The corresponding output intensity is

$$|\chi(\nu,\tau)|^2 = \begin{cases} (2T-|\tau|)^2\,\mathrm{sinc}^2\nu(2T-|\tau|); & |\tau| \le 2T \\ 0; & |\tau| \ge 2T. \end{cases} \tag{12}$$

For purposes of identification, it is instructive to examine the locus of points where the ambiguity func-