Mini Quiz: Professor Marks

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A binary asymmetric channel is characterized by the conditional probabilities

- $\Pr[Y = 1|X = 0] = q$
- $\Pr[Y = 0|X = 1] = p$

where both p and q are fixed probabilities. Assume $\Pr[X = 0] = x$ where x is a probability.

1. Calculate the mutual information of the channel.
2. Find the value of x that maximizes the mutual information.
3. Calculate the capacity of the channel.
4. Does the capacity of the binary symmetric channel result from the special case for $p = q$?

Solutions

1. Since $Pr[Y = 0] = r$ where $r = (1 - q)x + p(1 - x)$, we conclude $H(Y) = H_s(r)$. Also

$$H(Y|X) = H(Y|X = 0)\Pr(X = 0) + H(Y|X = 1)\Pr(X = 1) = H_s(q)x + H_s(p)(1 - x)$$

Thus

$$I(X, Y) = H(Y) - H(Y|X) = H_s(r) - H_s(q)x - H_s(p)(1 - x). \tag{1}$$

2. Differentiating gives

$$\frac{d}{dx}I(X, Y) = \frac{dr}{dx}\frac{d}{dr}H_s(r) - H_s(q) + H_s(p) = 0.$$

Since $\frac{dr}{dx} = 1 - p - q$ and

$$\frac{d}{dr}H_s(r) = \frac{d}{dr}\left(-r\log r - (1 - r)\log(1 - r)\right) = \log\left(\frac{1 - r}{r}\right),$$

we have, using nats instead of bits,

$$\frac{d}{dx}I(X, Y) = (1 - p - q)log\left(\frac{1 - r}{r}\right) - H_s(q) + H_s(p) = 0.$$

or

$$\log\left(\frac{1 - r}{r}\right) = \frac{H_s(q) - H_s(p)}{(1 - p - q)}$$

$$\frac{1 - r}{r} = e^{\frac{H_s(q) - H_s(p)}{(1 - p - q)}}$$

$$1 = r\left(e^{\frac{H_s(q) - H_s(p)}{(1 - p - q)}} + 1\right)$$

Thus

$$r^* = \left(e^{\frac{H_s(q)-H_s(p)}{(1-p-q)}} + 1\right)^{-1} = (1-q)x^* + p(1-x^*) = p + (1-q-p)x^* \tag{2}$$

and

$$x^* = \frac{r^*-p}{1-q-p} = \frac{\left(e^{\frac{H_s(q)-H_s(p)}{(1-p-q)}} + 1\right)^{-1} - p}{1-q-p} \tag{3}$$

3.

$$\max_x I(X,Y) = C = H_s(r^*) - H_s(q)x^* - H_s(p)(1-x^*)$$

4. When $p = q$, (1) becomes $I(X,Y) = H_s(r) - H_s(p)$. and the capacity is

$$C = H_s(r^*) - H_s(p) \tag{4}$$

This is maximized when $r^* = \frac{1}{2}$, so from (2), $\frac{1}{2} = p + (1-2p)x^*$ and, independent of p

$$x^* = \frac{\frac{1}{2}-p}{1-2p} = \frac{1}{2}.$$

and (4) becomes the same capacity as the binary symmetric channel.