# Experimental Measurement of Harmonic Coupling in TIPP Systems

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***Abstract*—Methodology is presented for experimental characterization of TIPP parameters.**

***Index Terms*—Harmonic Coupling,**



## I. Introduction

The time invariant periodicity preservation (TIPP) class of nonlinear systems [Original paper] has the property that periodic stimulation, $i(t)$ will result in a periodic response, $v(t)$, of the same period, $T$. For an operating point, the input-output relationship is expressed via their Fourier series coefficients. Temporally, the TIPP system is represented by the operator $\mathcal{Z}$ so that

$$v(t) = \mathcal{Z}\{i(t)\} \qquad (100)$$

Let the vector $\vec{\imath}$ contain the Fourier coefficients of $i(t)$. That is

$$i(t) = \sum_{m=-\infty}^{\infty} i_m \, e^{\frac{j2\pi mt}{T}}$$

where the Fourier series coefficients are

$$i_m = \frac{1}{T}\int_T i(t) e^{\frac{-j2\pi mt}{T}} \, dt$$

The vector $\vec{\imath}$ contains the Fourier series coefficients, i.e.

$$[\![\vec{\imath}]\!]_m = i_m$$

Likewise, let $\vec{v}$ contain the Fourier coefficients of $v(t)$. The temporal $\mathcal{Z}$ operator in (100) can now be recast in the Fourier domain as

$$\vec{v} = \mathbb{Z}\{\vec{\imath}\}$$

where $\mathbb{Z}$ is the frequency equivalent of the temporal operater, $\mathcal{Z}$. Expression in the frequency domain captures the cross harmonic coupling in the TIPP system.

Consider, then, a periodic perturbation $\Delta i(t)$ to the TIPP system and let

$$v_\Delta(t) = \mathcal{Z}\{\, i(t) + \Delta i(t)\}$$

Let $\Delta\vec{\imath}$ be the Fourier series coefficients of $\Delta i(t)$ and $\vec{v}_\Delta$ the coefficients of $v_\Delta(t)$. Then, equivalently,

$$\vec{v}_\Delta = \mathbb{Z}\{\vec{\imath} + \Delta\vec{\imath}\}$$

This response can be approximated by a *harmonically coupled affine* (HCA) system

$$\vec{v}_\Delta \approx \vec{v} + \Delta\vec{v}$$

where

$$\Delta\vec{v} = \vec{\nabla}\mathbb{Z}_{\vec{v}}(\vec{\imath})\Delta\vec{\imath} \qquad (200)$$

and the elements of the Jacobian are given by the gradient

$$\left[\vec{\nabla}\mathbb{Z}_{\vec{v}}(\vec{\imath})\right]_{nm} := \left[\!\!\left[\frac{\partial v_n}{\partial i_m}\right]\!\!\right]_{\vec{\imath}}$$

The HCA is a linearinzation of the cross-harmonic coupling of the TIPP system at the operating point. The TIPP paramenters, $\frac{\partial v_n}{\partial i_m}$, measure the coupling strength between the $m$th stimulus harmonic and the $n$th response harmonic.

## II. Laboratory Measuring the Elements of the Jacobia: The On-Frequency Method

Consider again the TIPP circuit in Figure 1(a). In Figure 1(b), the source is perturbed from a periodic signal with Fourier coefficients $\vec{\imath}$ to one with Fourier coefficients $\vec{\imath} + \Delta\vec{\imath}$. The reponse is then perturbed from $\vec{v}$ to $\vec{v} + \Delta\vec{v}$. In general, every element in $\Delta\vec{\imath}$ contributes to every element in the $\Delta\vec{v}$. We can determine the response to one frequency alone as is shown in Figure 5(c). The original signal is perturbed with a small increase at harmonic and the response is noted. The elements of the response are in the vector $\Delta_1\vec{v}$ whose $n$th element is $[\Delta_1\vec{v}]_n$. We estimate

$$\frac{\partial v_n}{\partial i_1} \simeq \frac{[\Delta_1\vec{v}]_n}{\Delta_1 i}$$

The second experiment is shown in Figure 1(c) where the quiescent stimulus is perturbed at a second harmonic and we estimate the partials $\frac{\partial v_n}{\partial i_2} \simeq \frac{[\Delta_2 \vec{v}]_n}{\Delta_2 i}$. Further experiments are conducted, as shown in Figure 1(d) and we catalog

$$[\vec{\nabla}\,\mathbb{Z}_{\vec{v}}(\vec{\imath})]_{nm} = \frac{\partial v_n}{\partial i_m}$$

$\simeq \frac{[\Delta_m \vec{v}]_n}{\Delta_m i}; 0 \leq n < \infty, -\infty \leq m \leq \infty.$

Given these values, we can estimate the perturbed response, $\Delta\vec{v}$, from a seource perturbation $\Delta\vec{\imath}$ using the HCA adjustment of



$$\Delta\vec{v} \simeq \vec{\nabla}\,\mathbb{Z}_{\vec{v}}(\vec{\imath})\Delta\vec{\imath}$$

or, equivalently, for $n \geq 0$,

$$\Delta v_n \simeq \sum_{m=-\infty}^{\infty} \frac{\partial v_n}{\partial i_m} \Delta i_m \qquad (400)$$

and, for $n < 0$, $\Delta v_n = \Delta v_{-n}^*$.

This sum can also be written in Wirtinger form as

$$\begin{aligned} \Delta v_n &\simeq \sum_{m=-\infty}^{-1} \frac{\partial v_n}{\partial i_m} \Delta i_m + \sum_{m=0}^{\infty} \frac{\partial v_n}{\partial i_m} \Delta i_m \\ &= \sum_{m=1}^{\infty} \frac{\partial v_n}{\partial i_{-m}} \Delta i_{-m} + \sum_{m=0}^{\infty} \frac{\partial v_n}{\partial i_m} \Delta i_m \\ &= \frac{\partial v_n}{\partial i_0} \Delta i_0 + \sum_{m=1}^{\infty} \left[\frac{\partial v_n}{\partial i_m} \Delta i_m + \frac{\partial v_n}{\partial i_m^*} \Delta i_m^*\right] \end{aligned}$$

## III. Using Real Signal Experiments

The analysis in the previous section is using complex perturbation of the form

$$\Delta_m i(t) = \varepsilon\, e^{j2\pi mt/T}$$

To use two real signals, we perform two experiments for every $m$. One perturbation uses the stimulus

$$\Delta_m^{\mathrm{c}}\ i(t) = \varepsilon \cos\left(\frac{2\pi mt}{T}\right)$$

and the other uses the quadrature signal

$$\Delta_m^{\mathrm{s}}\ i(t) = \varepsilon \sin\left(\frac{2\pi mt}{T}\right).$$

We can then evaluate

$$\Delta_m i(t) = \Delta_m^{\mathrm{c}}\ i(t) + j\Delta_m^{\mathrm{s}}\ i(t)$$

which is periodic with Fourier series coefficients

$$\Delta_m \vec{\imath} = \Delta_m^{\mathrm{c}}\ \vec{\imath} + j\Delta_m^{\mathrm{s}}\ \vec{\imath}$$

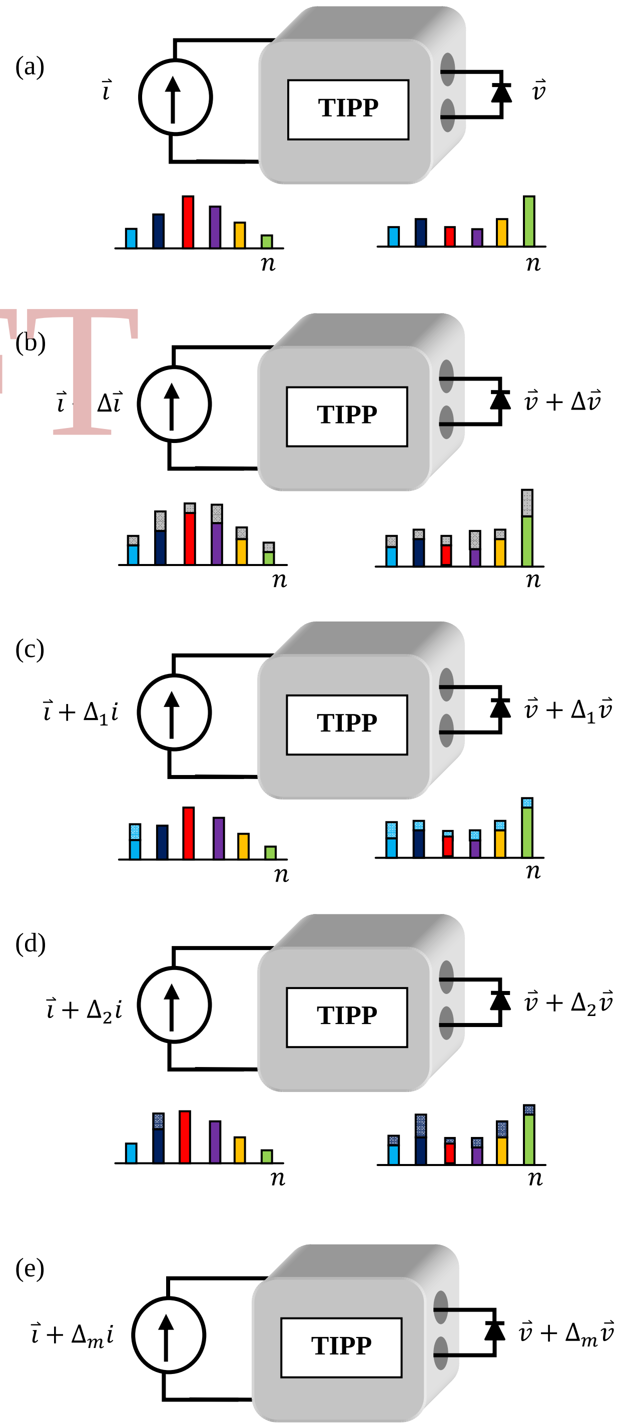


Figure 1. The on-frequency method of estimating TIPP parameters

where $\Delta_m^{\mathrm{c}}\ \vec{\imath}$ and $\Delta_m^{\mathrm{s}}\ \vec{\imath}$ are the Fourier coefficients of $\Delta_m^{\mathrm{c}}\ i(t)$ and $\Delta_m^{\mathrm{s}}\ i(t)$. Both $\Delta_m^{\mathrm{c}}\ \vec{\imath}$ and $\Delta_m^{\mathrm{s}}\ \vec{\imath}$ will have only two nonzero elements corresponding to the frequencies $\pm\frac{m}{T}$ Hertz. Let the response to $\Delta_m^{\mathrm{c}}\ i(t)$ and $\Delta_m^{\mathrm{s}}\ i(t)$ be the perturbations $\Delta_m^{\mathrm{c}}\ v(t)$ and $\Delta_m^{\mathrm{s}}\ v(t)$ with Fourier coefficients $\Delta_m^{\mathrm{c}}\vec{v}$ and $\Delta_m^{\mathrm{s}}\vec{v}$. Then, set

$$\Delta_m \vec{v} = \Delta_m^{\mathrm{c}}\vec{v} + j\Delta_m^{\mathrm{s}}\vec{v} \quad ; m \geq 0 \qquad (352)$$

When the DUT is real, we know that $\Delta_{-m}\vec{v} = \Delta_m \vec{v}^*$ so that the Fourier coefficients for negative $m$ can be found directly from the positive coefficients in (352). Finally, the elements of the Jacobian can be estimated using the on-frequency method using

$$\frac{\partial v_n}{\partial i_m} \simeq \frac{[\Delta_m \vec{v}\,]_n}{[\Delta_m \vec{\imath}]_m} \qquad (352a)$$

Only a finite number of experiments can be performed. Thus, we approximate

$$\Delta\, v_n \;\simeq \sum_{m=-M}^{M} \frac{[\Delta_m \vec{v}\,]_n}{[\Delta_m \vec{\imath}]_m}\, \Delta i_m \qquad (352b)$$

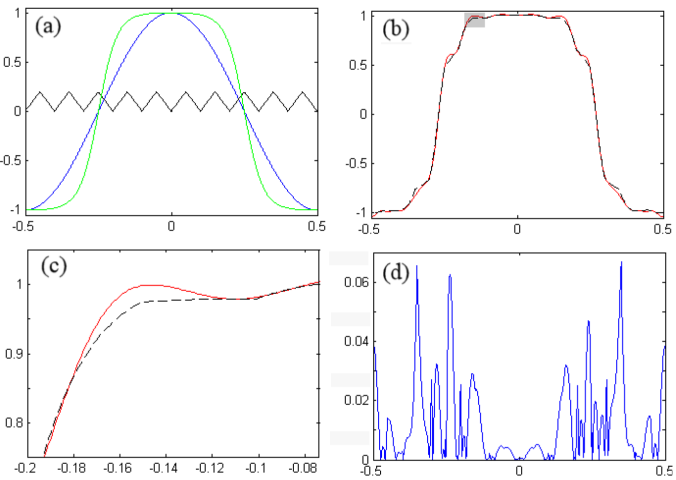


Figure 2: The on-frequency appoximation using, as an imput operating point, a unit amplitude cosine with period $T = 1$. The TIPS system is a memoryless nonlinear sigmoid given by (54). A value of $\vartheta = 0.4$ is used. The effect, as shown in (a), is to flatten the cosine. From this information, the Jacobian is estimated using the on-frequency method with $\varepsilon = 0.01$. Twenty positive frequency harmonics are used. The Jacobian is therefore $41 \times 41$.

A triangular wave perturbation is appied as shown in (a). The preturbation ranges between a minimum of zero and maximum of 0.2. The true TIPS system output for the perturbed cosinusoid is shown in (b) using a dashed line. The output computed using the JProx and is the solid line. Detail of the small shaded box in (b) is shown in (c). The maximum absolute error (the magnitude of the difference between the two curves in (b)) is shown in (d). The overall RMS error using the JProx is 0.0189.

A total of $2(M+1)$ experiments are therefore required.

1) *Example.* Let the TIPS system be characterized by a sigmoid parameterized by $\vartheta$.

$$\theta(i) = \frac{1+e^{-\vartheta}}{1+e^{-\vartheta i}} \qquad (54)$$

For a cosine input, as illustrated in Figure 2(a), the effect is to broaden the sinusoidal lobes as is done, for example, in AM conmpression. The cosine, which is the input operating point, is perturbed by the small triangular wave shown in Figure 2(a). The true output is favorably compared to the JProx augmented output operating point in Figure 2(b). Detail of the small shaded box in Figure 2(b) is shown in Figure 2(c). The absolute error between the actual output and the approximation is in Figure 2(d). Parameter details are given in the caption of Figure 2.

The number of harmonics used in the HCA effects the accuracy of the approximation. For the curves in Figure 2, $M = 20$. The resulting RMS error of the approximation is 0.0189. A plot of the RMS error versus $M$ for this problem is shown in Figure 3 As expected, the error decreases with $M$. Whether or not error generally decreases as a function of $M$, however, remains an open problem. Since a linear approximation is being used, however, we do know that the error, does not approach zero for large $M$.

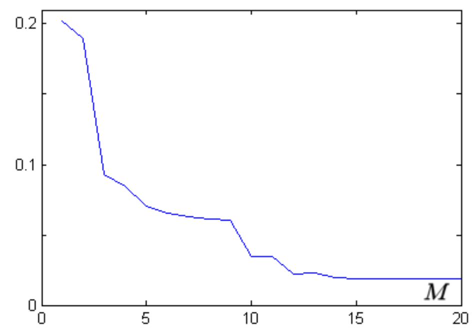


Figure 3: RMS error versus the harmonic count, $M$, for the problem considered in Figure 2.

1. Set $p = 1$, $\vec{\imath} = \vec{\imath}$, and $\vec{h} = \alpha_1 \vec{\imath}$.
2. $p + 1 \leftarrow p$. If $p > P$, stop. Else
3. $\vec{\imath} \leftarrow \vec{\imath} * \vec{\imath}$
4. $\vec{h} \leftarrow \vec{h} + p\alpha_p\, \vec{\imath}$
5. Go to Step 2

Step 3 can be executed knowing that the Fourier coefficients of $\vec{\imath}$ correspond to the temporal signal $i^p(t)$ in each loop.

The convolution in (800) can be expressed using Jacobians and TIPP parameters. For a convolution, the Jacobian is Toeplitz with parameters

$$[\![\vec{\nabla}\,\mathbb{Z}_{\vec{v}}(\vec{\imath}\,)]\!]_{nm} = \frac{\partial v_n}{\partial i_m} = [\![\vec{h}]\!]_{n-m}.$$

Then, in lieu of (800), we have the matrix-vector relationship in (200).