INTERPOLATION OF DISCRETE PERIODIC NONUNIFORM DECIMATION USING ALIAS UNRAVELING

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ABSTRACT

We consider the problem of signal restoration when P of every N samples in a discete time system are uniformly decimated. The degraded signal is an aliased form of the original signal. The aliasing can, in certain cases, be unraveled by application of multiplicative discrete time trigonometric polynomials followed by filtering. The filter output is the restored discrete time signal. Conditions required for this restoration are presented. The condition - and thus the noise sensitivity - of the restoration process is also analyzed.

1. INTRODUCTION

Discrete periodic nonuniform decimation occurs then a discrete signal, $f[n]$, is periodically set to zero over a specified interval. Let the period of the decimation be P and let $N < P$. Within a period, the first N values are known and the remaining $P - N$ are set to zero. This periodic decimation is repeated for all values of n. The periodically nonuniformly decimated signal is denoted by $g[n]$. Given that the original signal, $f[n]$, is bandlimited, the problem is, given $g[n]$ and the bandwidth B, to find, if possible, $f[n]$. This is illustrated in Figure 1 for $P = 5$ and $N = 3$. $N = P - 1$ corresponds to the well studied uniform periodic decimation [1]. The more general case where the first N of P values are known in each period is considered. The discrete periodic nonuniform decimation problem, though, can be made more general than this. In a period of $P = 5$, for example, the identity of the first, third and fifth values may be known and the others not. Interpolation in the case of these more general problems follows straightforwardly from the analysis to follow.

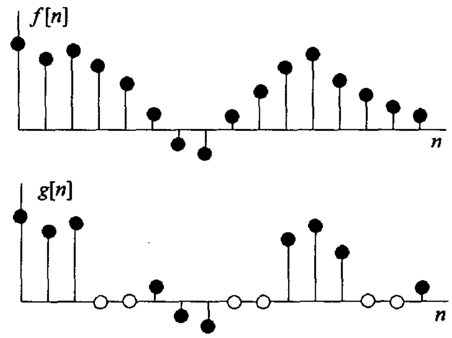

Fig. 1. Illustration of discrete periodic nonuniform decimation. Here, the period is $P = 5$. Each of the first $N = 3$ values are kept and the remaining $P - N = 2$ are discarded

2. PROBLEM DESCRIPTION

To generate the decimated signal, $g[n]$, define the discrete rectangular pulse train

$$r_P^N[n] = \sum_{p=-\infty}^{\infty} \Pi[pP \leq n < pP + N] \tag{1}$$

where $\Pi[n_- \leq n < n_+] = 1$ for $n_- \leq n < n_+$ and is otherwise zero. Then

$$g[n] = f[n] r_P^N[n]. \tag{2}$$

Our task is, when possible, to find $f[n]$ given its bandwidth and $g[n]$.

Taking the discrete time Fourier transform (DTFT)[1] of both sides of (2) gives the circular convolution

$$G(v) = F(v) * \left(R_P^N(v)\Pi(v)\right) \tag{3}$$

where $\Pi(v) =\equiv \Pi\left(-\frac{1}{2} \leq v \leq \frac{1}{2}\right)$ and $R_P^N(v)$ is the DTFT of $r_P^N[n]$. From (1), $R_P^N(v) = \sum_{p=-\infty}^{\infty} \sum_{n=pP}^{pP+N-1} e^{-j2\pi n v}$.

[1]The DTFT if a sequence $x[n]$ is $X(v) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi v}$.

In the n sum, let $m = n - pP$. Then

$$R_P^N(v) = \sum_{p=-\infty}^{\infty} e^{-j2\pi pPv} \sum_{m=0}^{N-1} e^{-j2\pi mv} \qquad (4)$$

The p sum is recognized as the Fourier series of $\sum_{k=-\infty}^{\infty} \delta(Pv$ $k)$. Using a geometric series, we can show

$$\sum_{m=0}^{N-1} e^{-j2\pi mv} = N e^{-j\pi(N-1)v} \mathrm{array}_N(v). \qquad (5)$$

where $\mathrm{array}_N(v) = \sin(\pi N v)/\sin(\pi v)$. Equation (4) therefore becomes

$$\begin{aligned} R_P^N(v) &= N\mathrm{comb}(Pv) e^{-j\pi(N-1)v} \mathrm{array}_N(v) \\ &= \sum_{p=-\infty}^{\infty} a_p \delta\left(v - \frac{p}{P}\right) \end{aligned}$$

where

$$a_p = \frac{N}{P} e^{-j\pi(N-1)p/P} \mathrm{array}_N\left(\frac{p}{P}\right). \qquad (6)$$

2.1. Degree of Aliasing

The function to be restored, $f[n]$, is assumed to have bandwidth $B < \frac{1}{2}$ so that $F(v) = F(v) r_{2B}(v)$ where $r_\alpha(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t-n}{\alpha}\right)$. The convolution replicates this periodic spectrum in a possibly aliased fashion. An example is shown in Figure 2 for $P = 3$. Since one spectrum from the right overlaps the zeroth order spectrum, the degree of aliasing is $M = 1$. In general, the lowest frequency component of the Mth spectrum is at $v = \frac{M}{P} - B$. We wish to determine the largest value of M that infringes on the interval of the zeroth order spectrum, *i.e.* the largest value of M such that $\frac{M}{P} - B \leq B$. The degree of aliasing follows as

$$M = < 2PB > \qquad (7)$$

where $< \zeta >$ denotes the largest integer not exceeding ζ.

2.2. Interpolation

For Mth order aliasing, as is the case for continuous sampling [2], a total of $2M$ overlapping spectra must be eliminated from their overlap of the zeroth order spectrum. We desire coefficients $\{\beta_q | -M \leq q \leq M\}$ such that superimposing $2M+1$ versions of $G(u)$ with various shifts illimiantes the aliasing specta and reconstructs the zeroth order spectrum exactly. We therefore seek the coefficients that solve

$$\left[\sum_{q=-M}^{M} \beta_q G\left(v - \frac{q}{P}\right)\right] = F(v) \ ; \ |v| \leq B. \qquad (8)$$

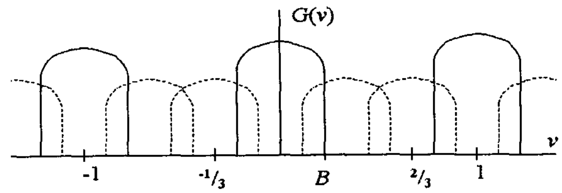

Fig. 2. Illustration of M = 1st order aliasing corresponding to $B = \frac{1}{4}$ and $P = 3$. The spectrum, $F(u)$, is shown with solid lines and the aliasing spectra with broken lines. From (7), $M = \langle \frac{3}{2} \rangle = 1$.

The β_q coefficients are solutions to the following system of $2M + 1$ linear equations.

$$\sum_{q=-M}^{M} \beta_q a_{p-q} = \begin{cases} 1 & ; \quad p = 0 \\ 0 & ; \quad 1 \leq |p| \leq M \end{cases} \qquad (9)$$

In matrix-vector notation, (9)is

$$\mathrm{A}[N,P]\vec{\beta} = \vec{\delta}_M \qquad (10)$$

where A is a $(2M+1) \times (2M+1)$ matrix with elements

$$\begin{aligned} (\mathrm{A}[N,P])_{nm} &= a_{n-m} \\ &= \tfrac{N}{P} e^{-j\pi(N-1)p/P} \mathrm{array}_N\left(\tfrac{p}{P}\right), \end{aligned} \qquad (11)$$

$\vec{\beta}$ is a vector of the β_q's, and $\vec{\delta}_M$ is a $2M+1$ vector of zeros except with a single "1" in the middle. The vector $\vec{\beta}$ is therefore equal to the middle column of A^{-1}.

Assuming these equations do not contain collinear terms, the values of $\{\beta_q | -M \leq q \leq M\}$ can be solved numerically. Since

$$\begin{aligned} f[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} F(v) e^{j2\pi vn} dv, \\ &= \int_{-B}^{B} F(v) e^{j2\pi vn} dv, \end{aligned} \qquad (12)$$

we have, from (8),

$$\begin{aligned} f[n] &= \int_{-B}^{B} F(v) e^{j2\pi vn} dv \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\sum_{q=-M}^{M} \beta_q G\left(v - \tfrac{q}{P}\right) \Pi\left(\tfrac{v}{2B}\right)\right] e^{j2\pi vn} dv \\ &= \sum_{q=-M}^{M} \beta_q \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[G\left(v - \tfrac{q}{P}\right) \Pi\left(\tfrac{v}{2B}\right)\right] e^{j2\pi vn} dv. \end{aligned} \qquad (13)$$

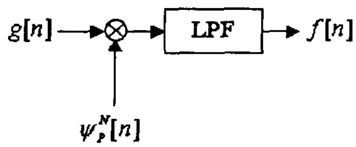

Fig. 3. Signal flow diagram for restoring a periodic nonuniform decimated discrete signal. The bandwidth of the low pass filter is B.

Since

$$\begin{aligned} f[n] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [X(v)H(v)]\, e^{j2\pi vn} dv \\ &= x[n] * h[n] \end{aligned}$$

where $*$ denotes discrete convolution, $x[n] \leftrightarrow X(v)$, and $h[n] \leftrightarrow H(v)$; and since

$$2B \text{ sinc}(2Bn) \leftrightarrow \Pi\left(\frac{v}{2B}\right),$$

and

$$g[n]e^{j2\pi nq/P} \leftrightarrow G\left(v - \frac{q}{P}\right), \tag{14}$$

(13) becomes

$$f[n] = \left[\Theta_M\left(\frac{n}{P}\right) g[n]\right] * 2B \text{ sinc}(2Bn) \tag{15}$$

where we define the trigonometric polynomial

$$\Theta_M(v) = \sum_{q=-M}^{M} \beta_q e^{j2\pi qv}.$$

Since, however, $g[n] = g[n]r_P^N[n]$, knowledge of $\Theta_M\left(\frac{n}{P}\right)$ is required only when $r_P^N[n] = 1$. Therefore, define the periodic function

$$\Psi_M\left(\frac{n}{P}\right) = \Theta_M\left(\frac{n}{P}\right) r_P^N[n]$$

and (15) becomes

$$f[n] = \left[g[n]\Psi_M\left(\frac{n}{P}\right)\right] * 2B \text{ sinc}(2Bn). \tag{16}$$

This procedure is illustrated in Figure(3).

2.3. The A[N, P] Matrix

The ability to solve the set of equations in (10) is dependent on the condition[2] of the matrix A[N, P]. We list two cases where the matrix is singular.

[2]The ratio of the largest to smallest eigen value magnitudes. The condition number is an indicator of the required computational accuracy. The condition number for singular matrices is ∞.

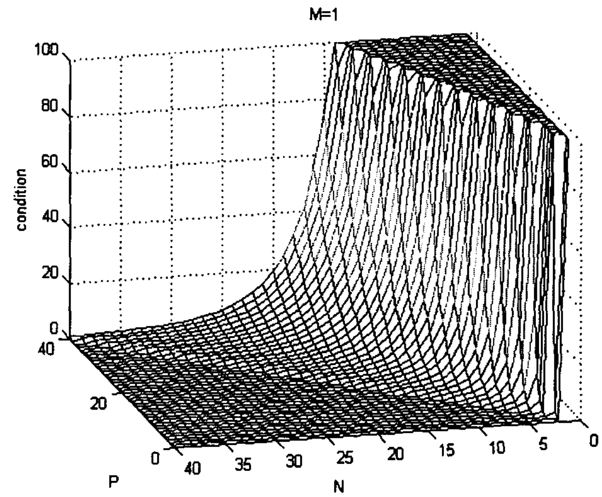

Fig. 4. A plot of matrix condition number for first order ($M = 1$) aliasing for various N and P. (Values for $N < P$ are shown set to zero).

1. The A[$1, P$] matrix is singular when $N = 1$. Thus, interpolation using (16) is not possible. To show this, we note that, since $\text{array}_1(x) = 1$, each element of the A[$1, P$] matrix in (11) is $a_{nm} = \frac{N}{P}$.
2. The matrix A[N, P] is singular when $B \geq \frac{1}{4}$. It is straightforward to show that the coefficient a_p in (6) is periodic with period P. The $2M$ shifts in (9) must not allow the an entire shift of a period into the matrix. Otherwise, two rows of A[N, P] will be identical. The matrix is therefore singular when $P \leq 2M$. Substituting the aliasing relation in (7) gives $P \leq 4BP$ from which the singularity condition $B \geq \frac{1}{4}$ results.

The condition number for A[N, P] can be computed directly. For example, when $M = 1$.

$P \Rightarrow$	2	3	4	5	6
$N = 1$	∞	∞	∞	∞	∞
2	∞	∞	∞	∞	∞
3	–	∞	4	13.1	34.0
4	–	–	∞	2.50	6.00

A plot of the condition number for $M = 1$ for P up to 40 is shown in Figure 4.

3. THE PERIODIC FUNCTIONS, $\Psi_M(V)$

The periodic functions, $\Psi_M(v)$, needed to interpolate the periodic nonuniformly decimated signal in (16) can be evaluated straightforwardly when the A[N, P] matrix is well conditioned. An example of the resulting

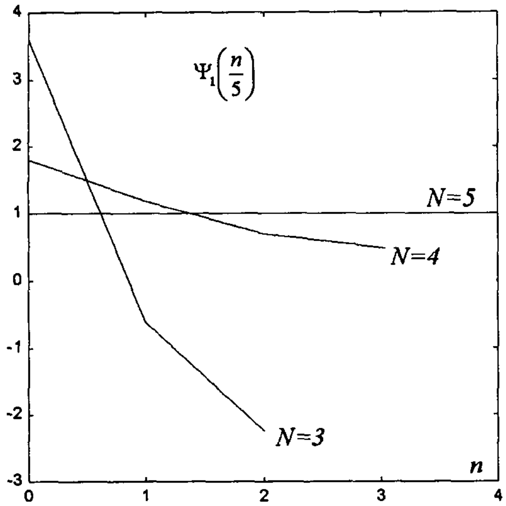

Fig. 5. Plots of $\Psi_M\left(\frac{p}{P}\right)$ for $M = 1$, $P = 5$ and $N = 3, 4, 5$.

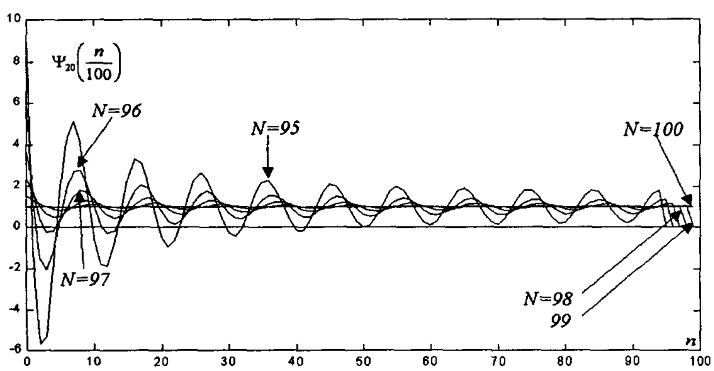

Fig. 6. Plots of $\Psi_M\left(\frac{p}{P}\right)$ for $M = 20$, $P = 100$ and $95 \le N \le 100$.

periodic functions are shown in Figures 5 and 6 for $(M,P) = (1,5)$ and $(M,P) = (20,100)$ respectively.

4. QUADRATURE VERSION

If $f[n]$ is real, its DTFT is conjugately symmetric. That is, $F(v) = F^*(-v)$. Therefore, if $f[n]$ is bandlimited, knowledge of $F(v)$ is only required on the interval $0 \le v \le B$. With reference to Figure 2, only the aliasing for positive v must be removed. There are M aliasing spectra to the right of the zeroth order spectra. In addition, there are spectra to the left of the positive frequency region of the zeroth order spectrum. The maximum frequency component of the $-m$th spectrum is $\frac{-m}{P} + B$. Let $-M_2$ be the minimum value of $-m$ for which this value is positive. Solving $\frac{-m}{P} + B > 0$ gives

$$M_2 = < BP > .$$

Note, for M even, $2M_2 = M$. Instead of the equation in (8), we desire to solve

$$\sum_{q=-M_2}^{M} \hat{\beta}_q G\left(v - \frac{q}{P}\right) = F(v) \; ; \; 0 \le v \le B. \qquad (17)$$

To find the $\hat{\beta}$ coefficients, instead of (9), we solve the following set of $M + M_2 + 1$ linear equations.

$$\sum_{q=-M_2}^{M} \hat{\beta}_q a_{p-q} = \begin{cases} 1 & ; \quad p = 0 \\ 0 & ; \quad -M_2 \le p \le -1 \text{ and } 1 \le p \le M \end{cases} \qquad (18)$$

5. NOTES

- For other parameters fixed, our conjecture is the restoration condition of alias unraveling increases with the degree of aliasing, M, and number of lost samples in a period, N, and decreases with period, P.
- There are other restoration procedures applicable to discrete periodic nonuniform decimation including the celebrated Papoulis-Gerchberg algorithm [2-3]. A comparison of the noise sensitivities of the alias unraveling procedure has yet to be performed. The alias unraveling approach, however, can be performed in real time using causal low pass filters in place of the ideal filters used in the theoretical development. The Papoulis-Gerchberg algorithm requires knowledge of the entire decimated signal before the iterative restoration algorithm can be applied.

6. REFERENCES

1. R. E. Crochiere & L. R. Rabiner Multirate Digital Signal Processing, Prentice Hall, 1996.
2. R.J. Marks II, Introduction to Shannon Sampling and Interpolation Theory, (Springer-Verlag, 1991).
3. R.J. Marks II, "Alternating Projections onto Convex Sets", in Deconvolution of Images and Spectra, edited by Peter A. Jansson, (Academic Press, San Diego, 1997), pp.476-501.