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Convergence of Howard's minimum-negativity-constraint extrapolation algorithm

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Howard's minimum-negativity-constraint extrapolation algorithm is shown to be a special case of signal recovery by means of alternating convex set projections. Previously derived results in this richly developed field of analysis [Appl. Opt. **25**, 1670 (1986); J. Opt. Soc. Am. **71**, 819 (1981)] are applied immediately to establish strong convergence for the extrapolation algorithm.

INTRODUCTION

Howard[1,2] proposed an iterative algorithm for extrapolating an interferogram when a portion of the interferogram is known. The algorithm is similar in flavor to the Papoulis–Gerchberg algorithm.[3–5] The only difference between the two is the constraint cast in the spectral domain: the Papoulis–Gerchberg algorithm uses the band-limitness constraint, whereas the Howard algorithm uses the nonnegativity constraint. Howard's algorithm was applied to experimental data and performed quite well.

In analog form, Howard's algorithm can be described as follows. Let $u(x) = u^*(-x)$ (where * denotes complex conjugation) denote an interferogram, and let $u_T(x) = u(x)$ over the interval $(-T, T)$ and $u_T(x) = 0$ elsewhere. Given that $u_T(x)$ is known, we desire to restore $u(x)$. Let $U(f)$ be the Fourier spectrum of $u(x)$. Howard's algorithm in analog form is then as follows:

Step 1. Set $N = 0$ and $u_N(x) = u_T(x)$.
Step 2. Generate $U_N(f)$.
Step 3. Set the positive values of $U_N(f)$ to zero.
Step 4. Subtract the inverse Fourier transform of this from $U_N(x)$.
Step 5. Set this signal to zero on the interval $(-T, T)$ and add $u_T(x)$.
Step 6. This signal is $u_{N+1}(x)$. Set $N = N + 1$. Go to step 2 and repeat.

In discrete form, the functions $U(f)$ and $u(x)$ are discrete sequences, and the Fourier transform is replaced by the discrete Fourier transform (DFT). In Howard's algorithm, the DFT coefficients $u_N(k)$'s are solved iteratively. In particular, they are formulated as an unknown vector in a matrix equation, which is then solved iteratively. In Appendix A, we show that the iterative matrix equation is an iterative form of the inverse DFT (IDFT) and that, in the absence of computational round-off noise, convergence is in one iteration. The iterative IDFT is incorporated into step 4 as an inner loop of the algorithm. When the iteration in the inner loop converges, the execution continues on the outer loop. One can, however, show analytically that in the absence of computational inexactness the inner loop converges in one iteration.[6]

Note that, algorithmically, we can replace steps 3 and 4 with the following steps:

Steps 3′ and 4′. Set the negative values of $U_N(f)$ to zero, and inverse transform.

In terms of implementation, however, the original procedure is generally more efficient in terms of the required number of operations.

Note that the algorithm is iterative imposes two constraints:

(a) The periodogram is equal to $u_T(x)$ on the interval $(-T, T)$.
(b) The nonnegativity constraint on the spectrum $U(f)$ (steps 3′ and 4′).

We show in this paper that the algorithm is a specific case of signal recovery by means of alternating convex set projections. If there is a set of functions that is consistent with both constraints, the algorithm converges to a result in this set. Otherwise convergence is to a function that satisfies constraints (a) and (b) in the minimum-mean-square sense. Equivalently, the steady-state solution after step 4 is a function that satisfies constraint (b) exactly and constraint (a) in the minimum-mean-square sense.

PROJECTION ONTO CONVEX SETS

The restoration algorithm described in the preceding section is a special case of alternating projections onto convex sets (POCS). In a signal space, a set of functions C is said to be convex if $\alpha u_1 + (1 - \alpha)u_2 \in C \; \forall u_1, u_2 \in C$, and $0 \le \alpha \le 1$.

The projection of any function $v(x)$ in the signal space

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onto a (closed) convex set results in that unique $u(x)$ function in C closest to $v(x)$ in the mean-square sense:

$$\|u(x) - v(x)\| = \inf_{w(x)\in C} \|w(x) - v(x)\|.$$

If $v(x) \in C$, then the projection operator is the identity operator. POCS can thus be viewed as a minimum-mean-square algorithm in which the constraints are convex.

Consider the case of M closed convex sets $C_1, C_2, \ldots, C_M$. Let P_i be the projection operator corresponding to the convex set C_i. The POCS algorithm can then be written in a composite form:

$$v_{N+1} = P_M P_{M-1} \ldots P_2 P_1 v_N,$$

where the subscript N is the index of the iteration. The composite projection operator alternatively projects onto the M sets. Let C_a be the intersection of the M convex sets:

$$C_a = \bigcap_{i=1}^{M} C_i,$$

where C_a is also convex. If $C_a \neq \varnothing$, then the sequence $\{v_i(x)\}$ converges to an element in C_a. Otherwise the sequence oscillates in a limit cycle among the convex sets.[7] If there are two nonintersecting sets, oscillation is between the points in each set closest to the other set.[8] Convergence can be accelerated by the use of relaxation.[9–11]

When the space in which the convex sets are subsummed is discrete and of finite dimension, convergence is strong (i.e., in the mean-square sense).[9,10]

Howard's algorithm consists of two convex constraints:

(a) C_i is the set of all functions equal to $u_T(x)$ on the interval $(-T, T)$. C_1 is a linear variety (a subspace from the origin). The corresponding projection operator for C_1 is

$$P_1 v(x) = \begin{cases} u_T(x) & x \in (-T, T) \\ v(x) & \text{otherwise} \end{cases}.$$

(b) C_2 is the set of all functions whose Fourier spectra are nonnegative. The corresponding projection operator for C_2 is

$$P_2 v(x) = F^{-1}\mu[V(\omega)],$$

where F denotes Fourier transformation, $V(\omega)$ is the Fourier spectrum of $v(x)$, and μ is the unit step. Clearly, the extrapolation algorithm under consideration is simply an iterative repetition of these projections. Howard's algorithm can be written as a composition of the two projection operations:

$$u_{N+1} = P_2 P_1 u_N.$$

If $C_1 \cap C_2 \neq \varnothing$, then alternating projections between them converge strongly in the mean-square sense to a point of intersection.

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