Detector Relative Efficiencies in the Presence of Laplace Noise

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The closed-form solution for Neyman-Pearson optimal detector performance for Laplace noise affords a rare opportunity for small and intermediate sample relative efficiency studies. Indeed, the Laplace noise solution is the only known closed-form description for a non-Gaussian optimal detector of the type considered. We illustrate numerically that, for stringent detector requirements, convergence of detector relative efficiencies to the corresponding asymptotic values can be quite slow. Three types of asymptotic efficiencies are considered in this comparison of the optimal, linear, and sign detectors.

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I. INTRODUCTION

In the analysis of detector performance of signals corrupted by noise, the noise is commonly assumed to be Gaussian. The assumption is often reasonable and usually results in a mathematically tractable analysis. There are, however, many instances where a non-Gaussian noise assumption is considered necessary [1–11]. In this paper, we analyze detector performance in the presence of Laplace (or two-sided exponential) noise. Because of its "heavy tail" behavior in comparison with Gaussian noise, Laplace noise has been suggested as a model for some types of impulsive noise (see [1, 2, 11]). In addition, except for the Gaussian case, *Laplace noise is the only known noise with a nowhere-zero probability density function for which a closed-form solution has been obtained for discrete, independent samples* [2]. These closed-form equations, however, can be computationally involved, with the result that so far, few comparative studies of the performance of the optimal detector versus nonoptimal detectors for Laplace noise have been attempted [11, 12]. With the aid of faster computation and a streamlined breakdown of the describing equations, we demonstrate that it is relatively straightforward to analyze the properties of the optimal detector, and to compare its performance with those of other nonoptimal detectors that may be more robust or more easily implemented [19, 13, pp. 74–93].

In this paper we compare the performance of the optimal detector with those of two other detectors: the linear detector and the sign detector (hard-limiter). The linear detector is optimal for Gaussian noise, and its properties have been well studied [14, 13, pp. 39–47]. The sign detector is a nonparametric detector that is nearly optimal for weak signals corrupted by Laplace noise [1]. We use a common method of comparing detectors, i.e., relative efficiency measures, in order to verify the validity of using such measures for small and intermediate sample sizes. This is of interest because for large sample sizes, the asymptotic relative efficiency (ARE), a limiting case of Pitman's relative efficiency (defined later), is widely used [1, 7, 15–17].[1] The ARE is easily calculated by an appeal to the central limit theorem, and although a valuable measure, it contains some deficiencies inherent in its definition which foster doubt in its application to small and intermediate sample sizes. Two other relative efficiencies are therefore presented and their asymptotic values are derived. The three measures present a more complete characterization of the detectors' relative performances. Numerical results are used to illustrate convergence to these asymptotic values.

II. PRELIMINARIES

The basic test for the detection of a constant-level, discrete-time signal S in the presence of noise is a choice between two hypotheses:

[1] ARE is defined in Section IIIA.

H_0: $x_i = n_i$

$$H_1: \quad x_i = S + n_i, \qquad \text{for } i = 1, \ldots, N. \tag{1}$$

We assume $S > 0$, and the noise samples $\{n_1, \ldots, n_N\}$ to be independent, identically distributed, real, with a Laplace pdf given by

$$f(x) \triangleq (\gamma_L/2) \exp\{-\gamma_L |x|\} \tag{2}$$

where λ_L is the Laplace parameter. Note that $\gamma_L = \sqrt{2}/\sigma$, where σ is the standard deviation of the Laplace noise.

Let α denote the probability of false alarm, i.e., the probability of deciding in favor of H_1 when no signal was sent, and let β be the probability of detection, i.e., the probability of deciding in favor of H_1 when a signal was actually sent.

A standard detector is illustrated in Fig. 1. The samples pass through a memoryless nonlinearity (MNL) denoted by $g(\cdot)$, are accumulated to form a test statistic t, and are compared with a fixed threshold T. The following decision is arrived at

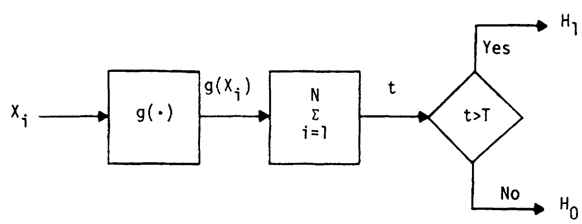


Fig. 1. Standard form of a detector.

$$t \begin{cases} > T, & \text{decide in favor of } H_1 \\ < T, & \text{decide in favor of } H_0 \end{cases}. \tag{3}$$

The detector types compared in this paper, i.e., the Neyman-Pearson optimal, linear, and sign detectors, each use the canonical form of Fig. 1 with a different MNL. In this section, we describe each detector and show how its performance is evaluated.

In detector analysis, the central limit theorem is often invoked on the test statistic t to estimate the detector's performance. These estimates are presented for each detector. Specifically, we compute the mean and variance of t and evaluate the resulting α and β estimates from use of these parameters in a Gaussian distribution.

A. The Neyman-Pearson Optimal Detector

The Neyman-Pearson detector is optimal in the sense that, for a fixed α (specified by the threshold), β is maximized. For Laplace noise, the nonlinearity for the optimal detector [1], illustrated in Fig. 2, is given by

$$g_{\mathrm{OPT}}(x) \triangleq \begin{cases} \gamma_D S, & \text{for } x > S \\ 2\gamma_D x - \gamma_D S, & \text{for } 0 \le x \le S \\ -\gamma_D S, & \text{for } x < 0. \end{cases} \tag{4}$$

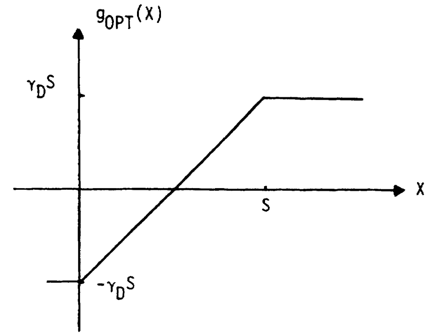


Fig. 2. MNL for the optimal detector.

The closed-form solution for the distribution function of the test statistic for the optimal detector with $\gamma = \gamma_D = \gamma_L$ has been derived earlier [2]. It is easily generalized without this constraint, by using the same procedure used in [2]. The result is

$$\begin{aligned} F_N^{(0)}(t) = {} & 2^{-N} \sum_{k=1}^{N} \binom{N}{k} \sum_{p=0}^{k} \binom{k}{p} (-1)^p \\ & \times \sum_{q=0}^{N-k} \binom{N-k}{q} \Bigg(\exp[-(p+q)\gamma_L S] \\ & - \exp\left[-\frac{\gamma_L}{2\gamma_D} (t + N\gamma_D S) \right] \\ & \times e_{k-1} \left\{ \frac{\gamma_L}{2\gamma_D} [t + (N - 2p - 2q)\gamma_D S] \right\} \Bigg) \\ & \times u[t + (N - 2p - 2q)\gamma_D S] \\ & + 2^{-N} \sum_{m=0}^{N} \binom{N}{m} \exp(-m\gamma_L S) \\ & \times u[t + (N - 2m)\gamma_D S] \end{aligned} \tag{5}$$

where $F_N^{(0)}(t)$ is the distribution function of the test statistic under H_0,

$$e_r(z) \triangleq \sum_{i=0}^{r} z^i/i!,$$

and

$$u(z) \triangleq \begin{cases} 0, & \text{for } z < 0 \\ 1, & \text{for } z \ge 0. \end{cases}$$

Also [2]

$$F_N^{(1)}(t) = 1 - F_N^{(0)}(-t) \tag{6}$$

where $F_N^{(1)}(t)$ is the distribution of the test statistic under hypothesis H_1.

Equation (5) reduces to the one in [2] if $\gamma = \gamma_D = \gamma_L$. Note that, without this constraint, the detector is still optimum in the Neyman-Pearson sense. With all other parameters fixed, the (α, β) coordinates follow the corresponding optimal receiver-operating-characteristic curve (ROC) as γ_L and/or γ_D is varied.

As mentioned in the introduction, the distribution functions in (5) and (6) furnish the only known closed-form solution for the Neyman-Pearson optimal detector for non-Gaussian noise with a density function that is nowhere-zero. The equations, however, are quite involved computationally. Some insight into an efficient evaluation of the expressions is given in the Appendix.

The performance of the optimal detector is determined directly from (5) and (6) via the relations

$$\alpha_{\text{opt}} = 1 - F_N^{(0)}(t) \tag{7}$$

$$\beta_{\text{opt}} = 1 - F_N^{(1)}(t) = F_N^{(0)}(-t). \tag{8}$$

The mean and variance of the test statistic are (see [2])

$$E_0\{t\} = -E_1\{t\} \triangleq m_{\text{opt}}$$
$$= N(\gamma_D/\gamma_L)[1-\exp(-\gamma_L S) - \gamma_L S] \tag{9}$$

and[2]

$$\text{var}_0\{t\} = \text{var}_1\{t\} \triangleq \sigma^2_{\text{opt}}$$
$$= N(\gamma_D/\gamma_L)^2[3-2\exp(-\gamma_L S)$$
$$-\exp(-2\gamma_L S) - 4\gamma_L S\exp(-\gamma_L S)]. \tag{10}$$

Using these as the Gaussian parameters, we obtain the central limit theorem's performance estimate for large N

$$\sigma^G_{\text{opt}} = 1 - \Phi[(T^G_{\text{opt}} - m_{\text{opt}})/\sigma_{\text{opt}}] \tag{11}$$

$$\beta^G_{\text{opt}} = 1 - \Phi[(T^G_{\text{opt}} + m_{\text{opt}})/\sigma_{\text{opt}}] \tag{12}$$

where $\Phi(x) = 1/(\sqrt{2\pi}) \int_{-\infty}^{x} \exp(-z^2/2)\,dz$ and T^G_{opt} is the threshold of the Gaussian approximation to the optimal detector. (Each detector's threshold is set at a different level for equivalent performance).

B. The Linear Detector

For the linear detector, the MNL is given by

$$g_{\text{lin}}(x) \triangleq x. \tag{13}$$

The distribution functions $G_N^{(0)}(t)$ and $G_N^{(1)}(t)$ under hypotheses H_0 and H_1 respectively for the linear detector have been derived [2] and are

$$G_N^{(0)}(t) = \begin{cases} \frac{1}{2} + \sum_{k=0}^{N-1} 2^{-(N+k)} \binom{N+k-1}{k} \\ \quad \times [1-\exp(-\gamma_L t)\, e_{N-k-1}(\gamma_L t)], \\ \qquad\qquad \text{for } t \geq 0 \\ 1 - G_N^{(0)}(-t), \qquad \text{for } t < 0 \end{cases} \tag{14}$$

$$G_N^{(1)}(t) = G_N^{(0)}(t-NS). \tag{15}$$

Therefore, we have

$$\alpha_{\text{lin}} = 1 - G_N^{(0)}(t) \tag{16}$$

$$\beta_{\text{lin}} = 1 - G_N^{(1)}(t) = 1 - G_N^{(0)}(t-NS). \tag{17}$$

The mean and variance of the test statistic here are

$$m_{\text{lin}} = NS$$

$$\sigma^2_{\text{lin}} = 2N/\gamma_L^2.$$

The corresponding central limit theorem approximations of the detector performance measures for large N are

$$\alpha^G_{\text{lin}} = 1 - \Phi[T^G_{\text{lin}}/\sigma_{\text{lin}}] \tag{18}$$

$$\beta^G_{\text{lin}} = 1 - \Phi[(T^G_{\text{lin}} - m_{\text{lin}})/\sigma_{\text{lin}}] \tag{19}$$

where T^G_{lin} is the threshold of the Gaussian approximation to the linear detector.

C. The Sign Detector

The sign detector has the nonlinearity

$$g_{\text{sgn}}(x) = \begin{cases} 1, & \text{for } x > 0 \\ -1, & \text{for } x < 0 \end{cases}.$$

The performance of the sign detector has been derived (see [18, 19, 13, pp. 47–51].) Let

$$p = \Pr[X_i > 0|H_1] = 1 - (\tfrac{1}{2})\exp(-\gamma_L S) \tag{20}$$

and $q = 1-p$. Then

$$\beta_{\text{sgn}} = \sum_{k=c+1}^{N} \binom{N}{k} p^k q^{N-k} \tag{21}$$

where c is the largest integer satisfying

$$\alpha_{\text{sgn}} \leq \sum_{k=c+1}^{N} \binom{N}{k} (\tfrac{1}{2})^N, \tag{22}$$

i.e., $c+1$ is the threshold T for the sign detector. The Gaussian approximation for the performance measures here is (from [20, p. 66] see also [14])

$$\alpha^G_{\text{sgn}} = 1 - \Phi[(T^G_{\text{sgn}} - N/2)/(\sqrt{N}/2)] \tag{23}$$

$$\beta^G_{\text{sgn}} = 1 - \Phi[(T^G_{\text{sgn}} - Np)/\sqrt{Npq}] \tag{24}$$

where T^G_{sgn} is the threshold of the Gaussian approximation to the sign detector. This approximation surprisingly fails to give correct results for some performance measures, even asymptotically, as it is shown later.

III. RELATIVE EFFICIENCY PERFORMANCE

A. General

Performance comparisons among the three detectors can be measured in many ways. Perhaps the most widely used technique is due to Pitman, whose definition of relative efficiency is (see [11, 13, pp. 31–36])

$$\text{RE}_{2,1} \triangleq N_1(\alpha,\beta,S)/N_2(\alpha,\beta,S) \tag{25}$$

[2]Note that $\text{var}_0\{t_{\text{opt}}\} = N\,\text{var}_0\{g_{\text{opt}}(X)\}$ rather than $N^2\,\text{var}_0\{g_{\text{opt}}(X)\}$ as given in [2]. Hence [2, Table 1] is invalid.

where $N_1(\alpha,\beta,S)$ is the number of samples detector one requires to achieve the given α and β at the signal level S, while $N_2(\alpha,\beta,S)$ is the number of samples required by detector two for the same α, β, and S. This can be a problem for small sample sizes, since it is usually not possible for a fixed α, β, and S to find both the N_1 and the N_2 that realize these exactly [13, p. 32]. Either interpolation, or upper and lower bounds such as employed in [11], must be used. Also, in general, the RE is difficult to calculate. In view of all this, the most commonly used measure is a limit (if it exists) of RE. Asymptotic relative efficiency is defined as

$$\mathrm{ARE}^S_{2,1} \triangleq \lim_{\substack{N_1\to\infty\\ N_2\to\infty\\ S\to 0}} N_1(\alpha,\beta,S)/N_2(\alpha,\beta,S) \tag{26}$$

while maintaining a constant α and β.

A common misconception about the ARE^S is that it provides a "fairly" accurate approximation to the relative efficiency for moderate sample sizes [21]. Indeed, the results of Michalsky et al. [14] show examples where the RE for nonoptimal detectors converges slowly or unexpectedly to the ARE. The measure also assumes a vanishingly small signal.

Various alternative measures of efficiency have been proposed in the literature [15, 16, 22, 27]. For a summary, see [13, pp. 31–33, 87–93]. Apart from the ARE^S, we use two additional measures ARE^α and ARE^β. These measures have the advantage of maintaining a fixed, finite signal strength S while N increases, which may be more realistic than a vanishingly small S in many applications. We define

$$\mathrm{ARE}^\alpha_{2,1} \triangleq \lim_{\substack{N_1\to\infty\\ N_2\to\infty\\ \alpha\to 0}} N_1(\alpha,\beta,S)/N_2(\alpha,\beta,S) \tag{27}$$

maintaining a fixed β and S. Similarly,

$$\mathrm{ARE}^\beta_{2,1} \triangleq \lim_{\substack{N_1\to\infty\\ N_2\to\infty\\ \beta\to 1}} N_1(\alpha,\beta,S)/N_2(\alpha,\beta,S) \tag{28}$$

while α and S are fixed.

We now derive expressions for ARE^α and ARE^β under the assumption that $g(X_i)$ has a distribution function that allows invocation of the central limit theorem on the test statistic t.

First, for ARE^α, let $\beta^G_{\mathrm{opt}} = \beta^G_{\mathrm{lin}} = \beta^G_{\mathrm{sgn}} = \beta_0$. Then

$$T^G_{\mathrm{opt}} = \sigma_{\mathrm{opt}}\,\nu_\beta - m_{\mathrm{opt}} \tag{29}$$

where

$$\nu_\beta \triangleq \Phi^{-1}[1 - \beta_0]. \tag{30}$$

Substituting into (11) gives

$$\alpha^G_{\mathrm{opt}} = 1 - \Phi[\nu_\beta - 2m_{\mathrm{opt}}/\sigma_{\mathrm{opt}}]. \tag{31}$$

Similarly, for the linear and sign detectors, using (18), (19), (23), and (24), we have

$$\alpha^G_{\mathrm{lin}} = 1 - \Phi[\nu_\beta + m_{\mathrm{lin}}/\sigma_{\mathrm{lin}}] \tag{32}$$

$$\alpha^G_{\mathrm{sgn}} = 1 - \Phi[2\sqrt{pq}\,\nu_\beta + \sqrt{N_{\mathrm{sgn}}}\,(2p-1)]. \tag{33}$$

Let

$$W(S) \triangleq m^2_{\mathrm{opt}}/\sigma^2_{\mathrm{opt}}. \tag{34}$$

From (9), (10), and (31), we define C_{opt} via

$$C_{\mathrm{opt}}\sqrt{N_{\mathrm{opt}}} \triangleq 2m_{\mathrm{opt}}/\sigma_{\mathrm{opt}}. \tag{35}$$

Similarly

$$C_{\mathrm{lin}}\sqrt{N_{\mathrm{lin}}} \triangleq m_{\mathrm{lin}}/\sigma_{\mathrm{lin}}. \tag{36}$$

Equating α^G_{opt} to α^G_{lin}, we obtain

$$-C_{\mathrm{opt}}\sqrt{N_{\mathrm{opt}}} = C_{\mathrm{lin}}\sqrt{N_{\mathrm{lin}}}$$

and

$$\mathrm{ARE}^\alpha_{\mathrm{opt,lin}} = \lim_{\substack{N_{\mathrm{opt}}\to\infty\\ N_{\mathrm{lin}}\to\infty}} \frac{N_{\mathrm{lin}}}{N_{\mathrm{opt}}} = \left(\frac{C_{\mathrm{opt}}}{C_{\mathrm{lin}}}\right)^2. \tag{37}$$

A similar manipulation yields

$$\sqrt{\frac{N_{\mathrm{sgn}}}{N_{\mathrm{opt}}}} = \frac{1}{(2p-1)}\left[\frac{(1-2\sqrt{pq})\nu_\beta}{\sqrt{N_{\mathrm{opt}}}} - C_{\mathrm{opt}}\right]. \tag{38}$$

Taking the limit as $N_1, N_2 \to \infty$, we get

$$\mathrm{ARE}^\alpha_{\mathrm{opt,sgn}} = \frac{C^2_{\mathrm{opt}}}{(2p-1)^2} = \frac{4W(S)}{(2p-1)^2}. \tag{39}$$

The foregoing treatment can be repeated for ARE^β. Setting $\alpha^G_{\mathrm{opt}} = \alpha^G_{\mathrm{lin}} = \alpha^G_{\mathrm{sgn}} = \alpha_0$, and defining $\nu_\alpha = 1 - \Phi^{-1}[1-\alpha_0]$, we obtain

$$\beta^G_{\mathrm{opt}} = 1 - \Phi[\nu_\alpha + 2m_{\mathrm{opt}}/\sigma_{\mathrm{opt}}] \tag{40}$$

$$\beta^G_{\mathrm{lin}} = 1 - \Phi[\nu_\alpha - m_{\mathrm{lin}}/\sigma_{\mathrm{lin}}] \tag{41}$$

$$\beta^G_{\mathrm{sgn}} = 1 - \Phi\{[\nu_\alpha - \sqrt{N_{\mathrm{sgn}}}(2p-1)]/2\sqrt{pq}\}. \tag{42}$$

After a straightforward calculation, we get

$$\mathrm{ARE}^\beta_{\mathrm{opt,lin}} = \mathrm{ARE}^\alpha_{\mathrm{opt,lin}} \tag{43}$$

and

$$\sqrt{\frac{N_{\mathrm{sgn}}}{N_{\mathrm{opt}}}} = \frac{-2\sqrt{pq}}{(2p-1)}\left[C_{\mathrm{opt}} + \frac{\nu_\alpha}{\sqrt{N_{\mathrm{opt}}}}\left(1 - \frac{1}{2\sqrt{pq}}\right)\right]. \tag{44}$$

Therefore

$$\mathrm{ARE}^\beta_{\mathrm{opt,sgn}} = \frac{16pqW(S)}{(2p-1)^2} = 4pq\,\mathrm{ARE}^\alpha_{\mathrm{opt,sgn}}. \tag{45}$$

It is interesting to note that (37), (39), (43), and (45) do not depend on ν_α and ν_β, but instead, depend only on S. This permits an analysis of detector performance using a nonzero signal strength.

In order to compare the rate with which the RE^S converges to the ARE^S for different values of α and β, it is easiest to find N_{opt} as a function of S from the normal approximation. From (11) and (12), we get

$$N_{opt} = (\nu_\beta - \nu_\alpha)^2/4W(S). \tag{46}$$

Then N_{lin} and N_{sgn} can be obtained by using (37) and (43) with either (39) or (45). Using S as an independent variable eliminates the need for iterating to find the values for N_{lin} and N_{opt}.

B. For Laplace Noise

For the specific case of Laplace noise, others [1, 13, p. 81] have shown that

$$ARE^S_{opt,lin} = 2 \tag{47}$$

$$ARE^S_{opt,sgn} = 1. \tag{48}$$

Equation (48) is not surprising since, for $S \to 0$, the nonlinearity for the optimal detector approaches that of the sign detector.

From (9), (10), and (34), we have for Laplace noise

$$W(S) = [1 - \exp(-\gamma_L S) - \gamma_L S]^2 / [3 - 2\exp(-\gamma_L S) - \exp(-2\gamma_L S) - 4\gamma_L S \exp(-\gamma_L S)]. \tag{49}$$

Thus, (37) becomes

$$ARE^\alpha_{opt,lin} = \frac{8W(S)}{S^2\gamma_L^2} = ARE^\beta_{opt,lin} \tag{50}$$

where we have used (43). The values of $ARE^{\alpha,\beta}_{opt,sgn}$ follow from (39) and (45) with $W(S)$ in (49) and the p in (20).

Finally, we note that the threshold of the optimal detector converges to a finite value for the ARE^S. From (11) and (12)

$$T^G_{opt} = \sigma_{opt}\,\nu_\alpha + m_{opt} = \sigma_{opt}\,\nu_\beta - m_{opt}.$$

Thus

$$T^G_{opt} = \sigma_{opt}(\nu_\alpha + \nu_\beta)/2 \tag{51}$$

$$= m_{opt}(\nu_\alpha + \nu_\beta)/(\nu_\beta - \nu_\alpha). \tag{52}$$

Squaring (51) and dividing by (52), we obtain

$$T^G_{opt} = \frac{\sigma^2_{opt}}{m_{opt}} \frac{\nu_\beta^2 - \nu_\alpha^2}{4}. \tag{53}$$

However,

$$\frac{\sigma^2_{opt}}{m_{opt}} = \frac{3 - 2\exp(-\gamma_L S) - \exp(-2\gamma_L S) - 4\gamma_L S\exp(-\gamma_L S)}{1 - \exp(\gamma_L S) - \gamma_L S}. \tag{54}$$

Applying l'Hopital's rule twice gives

$$\lim_{\gamma_L S \to 0} T^G_{opt} = (\nu_\alpha^2 - \nu_\beta^2)/2. \tag{55}$$

It is easily shown that except for the above, the threshold of all three detectors and their Gaussian equivalents under ARE^S, ARE^α, and ARE^β do not have a corresponding finite limit.

IV. NUMERICAL RESULTS

A. Results for RE^S

Miller and Thomas [11] have obtained results under the restriction that $\alpha = 1 - \beta$. Michalski, Wise, and Poor [14] have evaluated $RE^S_{lin,sgn}$ by using exact formulas and have numerically illustrated its convergence to the corresponding Gaussian approximation, again under the restriction above. Our treatment imposes no restriction on α and β and deals with the RE with respect to the optimal detector. Also, we present results on the effect of the signal strength and the threshold on N_{opt}.

Figs. 3(a) through 3(f) show the values for $RE^S_{opt,lin}$ and $RE^S_{opt,sgn}$ for values of N_{opt} between 1 and 50. These curves are plotted for various values of α and β, and both the exact values and the Gaussian approximations are displayed. The computed values are linearly connected for easier viewing.

For the linear detector, it is seen that the convergence of the exact $RE^S_{opt,lin}$ to its Gaussian counterpart and of both to the asymptotic value of 2 is faster for the case where $\alpha = 0.1$ and $\beta = 0.9$ than it is when $\alpha = 0.00001$ and $\beta = 0.99999$. This is true in general, as was noted earlier [11, 14]. Specifically, the more stringent the restrictions are on α and β, the slower is the convergence of both the exact and the Gaussian $RE^S_{opt,lin}$ to the $ARE^S_{opt,lin}$, and to each other. This is evident on examining Fig. 3(a), (c), and (e), and also on Fig. 4, which shows $RE^S_{opt,lin}$ and $RE^S_{opt,sgn}$ for relatively large values of N_{opt} for the Gaussian approximation alone. These curves, as before, are plotted for various values of α and β. Note that the plots are semilog here.

For the sign detector, the value of $RE^S_{opt,sgn}$ for small values of N_{opt} is quite high, and increases further as more stringent requirements are imposed on α and β. At $N_{opt} = 1$ for $\alpha = 0.1$ and $\beta = 0.9$, the RE is about 14 [Fig. 3(b)], while for $\alpha = 0.1$ and $\beta = 0.99999$, it is about 90 [Fig. 3(d)], and for $\alpha = 0.00001$ and $\beta = 0.99999$, it shoots up to about 160 [Fig. 3(f)]. The convergence to the asympotic value of one is also slower, as with the linear detector.

B. Dependence of S on N_{opt} for RE^S

Fig. 5 is a plot $\log_{10}S$ versus $\log_{10}N_{opt}$, using

$$N_{opt} = C_{\alpha\beta}/W(S). \tag{56}$$

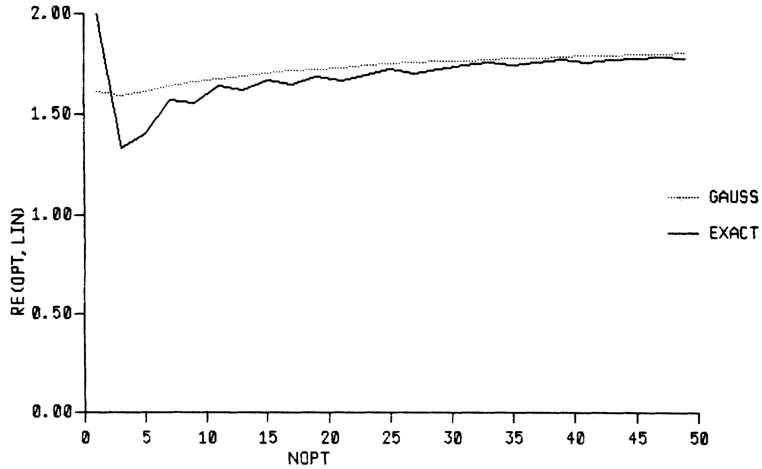
with $C_{\alpha\beta} = 1$. The plot is a straight line. Comparing (56) with (46), we let

$$C_{\alpha\beta} = (\nu_\beta - \nu_\alpha)^2/4 \tag{57}$$

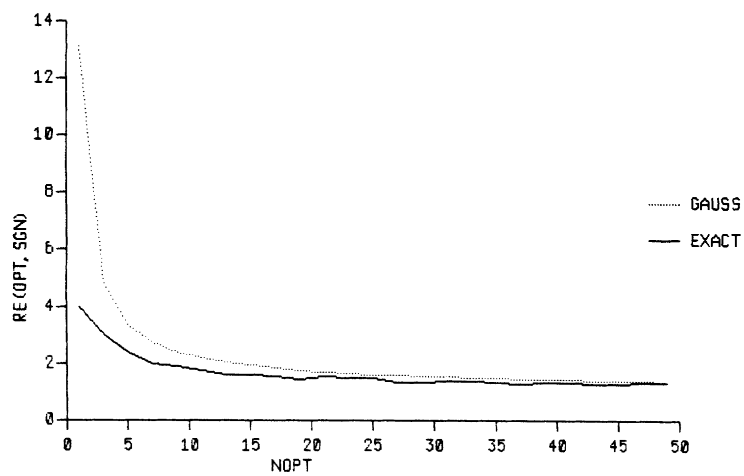
which is a constant for a fixed α and β. Varying the value of $C_{\alpha\beta}$ shifts the curve in Fig. 5 vertically, but does not change its slope. Therefore, for a fixed S, and specified α and β, we can find approximately the N_{opt} needed to realize the given α and β. This approximation is good for a small S and a large N_{opt}. In fact, using the power series for the exponential function, and neglecting the third and higher order terms for $S << 1$, we can show

$$N_{opt} \simeq (\nu_\beta - \nu_\alpha)^2/S^2 \quad \text{for} \quad S << 1. \tag{58}$$

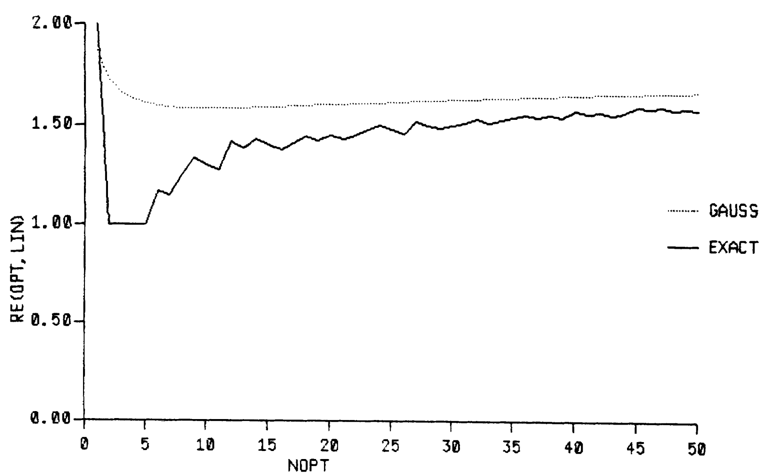
For small N_{opt}, the exact and Gaussian approximation values of S needed to realize a given α and β for a fixed



(a)

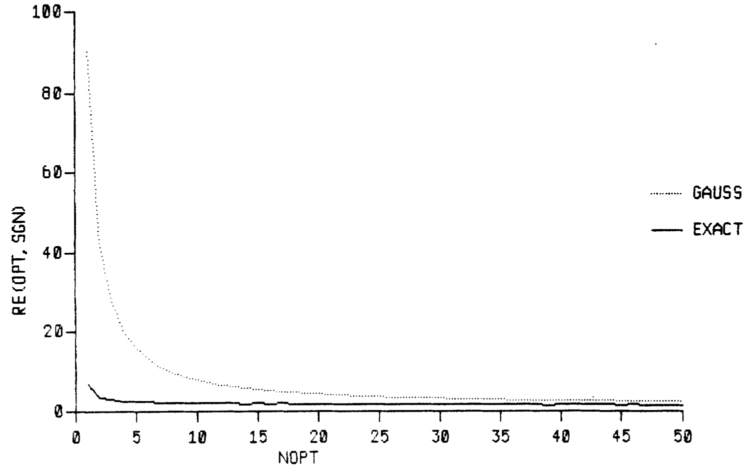


(b)

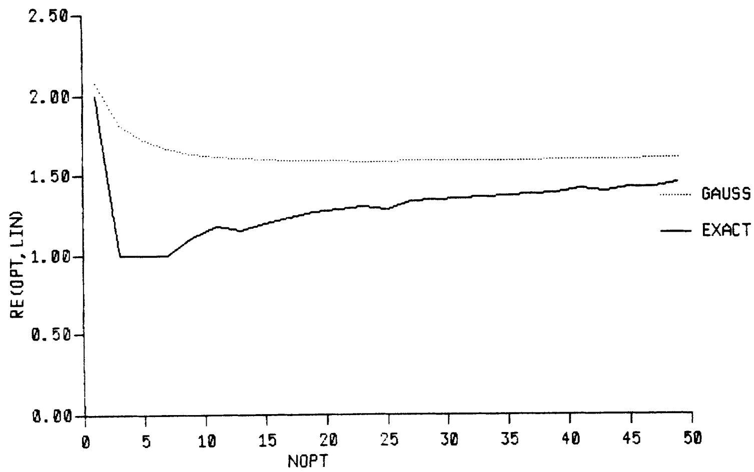


(c)

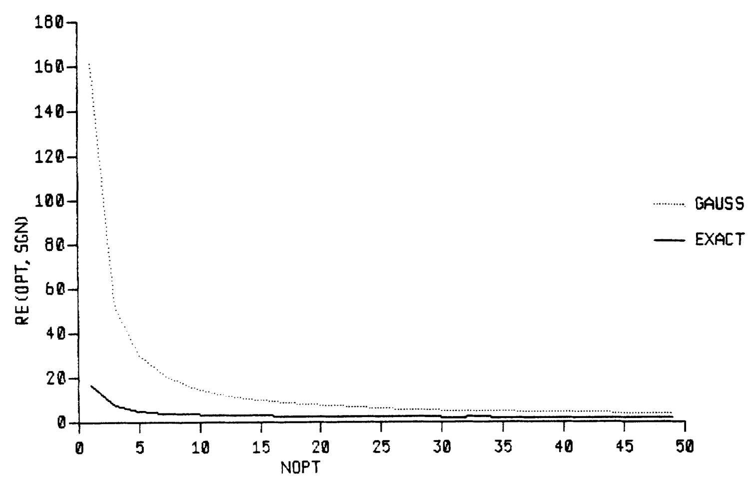
Fig. 3. RE^S for small N_{opt}. (a) $RE_{opt,lin}$ $\alpha = 0.1$, $\beta = 0.9$. (b) $RE_{opt,sgn}$ $\alpha = 0.1$, $\beta = 0.9$. (c) $RE_{opt,lin}$ $\alpha = 0.1$, $\beta = 0.99999$.



(d)



(e)



(f)

Fig. 3 (con't.) (d) $RE_{opt,sgn}$ $\alpha = 0.1$, $\beta = 0.99999$. (e) $RE_{opt,lin}$ $\alpha = 0.00001$, $\beta = 0.99999$. (f) $RE_{opt,sgn}$ $\alpha = 0.00001$, $\beta = 0.99999$.

N_{opt} are plotted in Fig. 6(a) and 6(b). For $\alpha = 0.1$ and $\beta = 0.9$, the agreement between the two is remarkable. As before, the more stringent the requirements are on α and β, the slower is the convergence of the Gaussian S to the exact S. The Gaussian approximation for the optimal detector is somewhat optimistic here.

C. The Threshold of the Optimal Detector

The exact threshold T_0 and the Gaussian approximation T_0^G using (11), (12), and (46) are plotted versus N_{opt} in Fig. 7, while T_0^G and the limiting value of the threshold T_{lim} using (55) are plotted versus $\log_{10} N_{opt}$

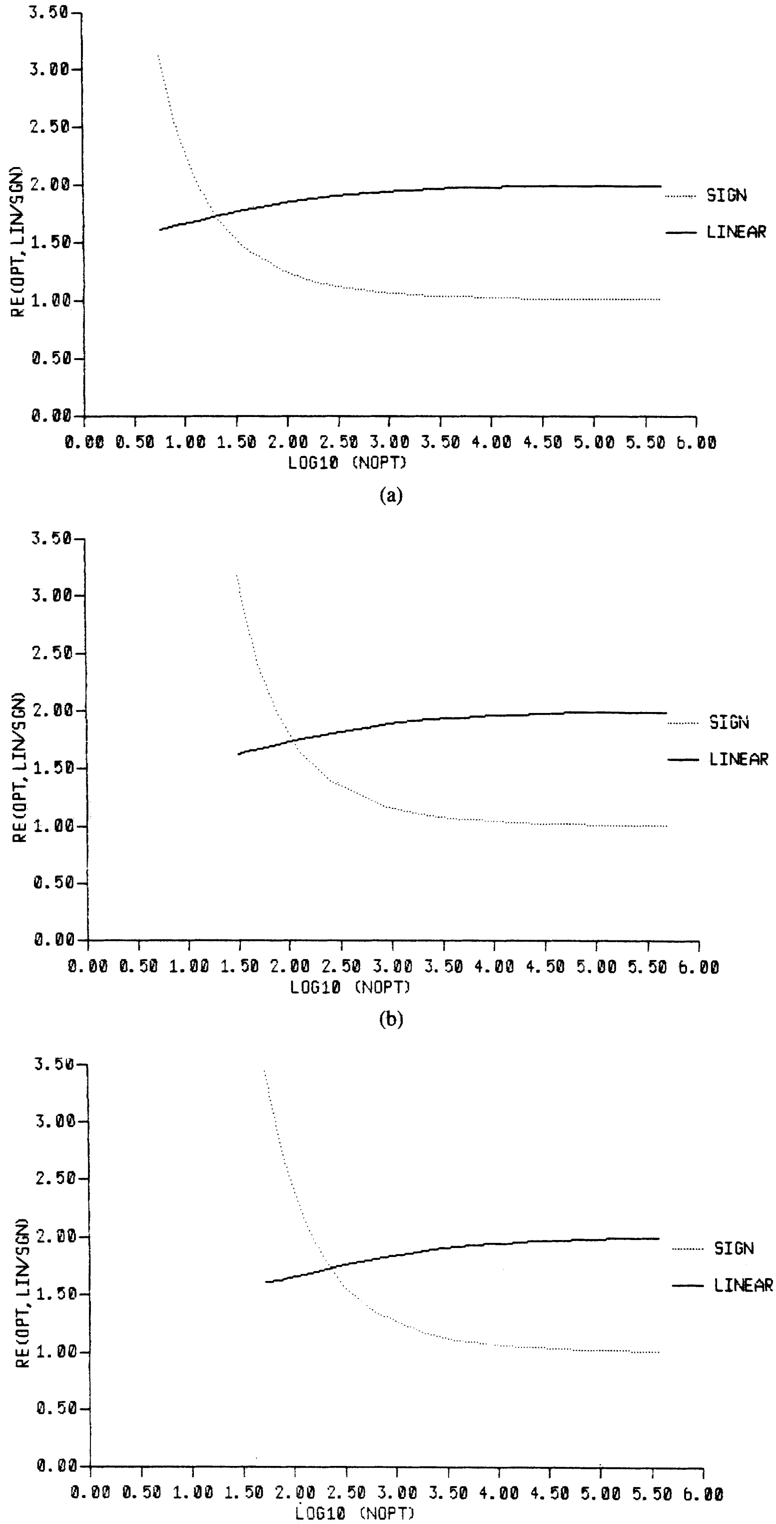


Fig. 4. RE^S for large N_{opt}. (a) $\alpha = 0.1$, $\beta = 0.9$. (b) $\alpha = 0.1$, $\beta = 0.99999$. (c) $\alpha = 0.00001$, $\beta = 0.99999$.

for large N_{opt} in Fig. 8. For the case when $\alpha = 1 - \beta$, T_{lim} is zero. We are not able to make any generalizations about the rates of convergence of T_0 and T_0^G to T_{lim} and to each other with respect to the severity of restrictions imposed on α and β, as was done previously for RE^S. However, it is interesting to note that T_0^G is very close to T_{lim} for $\log_{10} N_{opt} > 4$, as seen in Fig. 8. A knowledge of T_{lim} may be useful in setting the threshold of an optimal detector (see Fig. 1) when N_{opt} is either large, or is not specified beforehand.

D. Results for RE^α and RE^β

Fig. 9 is a plot of $ARE^\alpha_{opt,lin}$ as a function of S using (50). Note that the curve attains a minimum of about 1.58 at $S \simeq 1.97$ and then slowly rises to its limit value

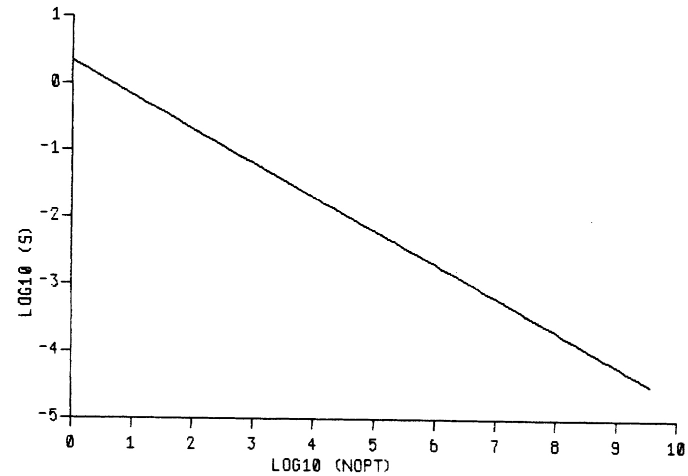
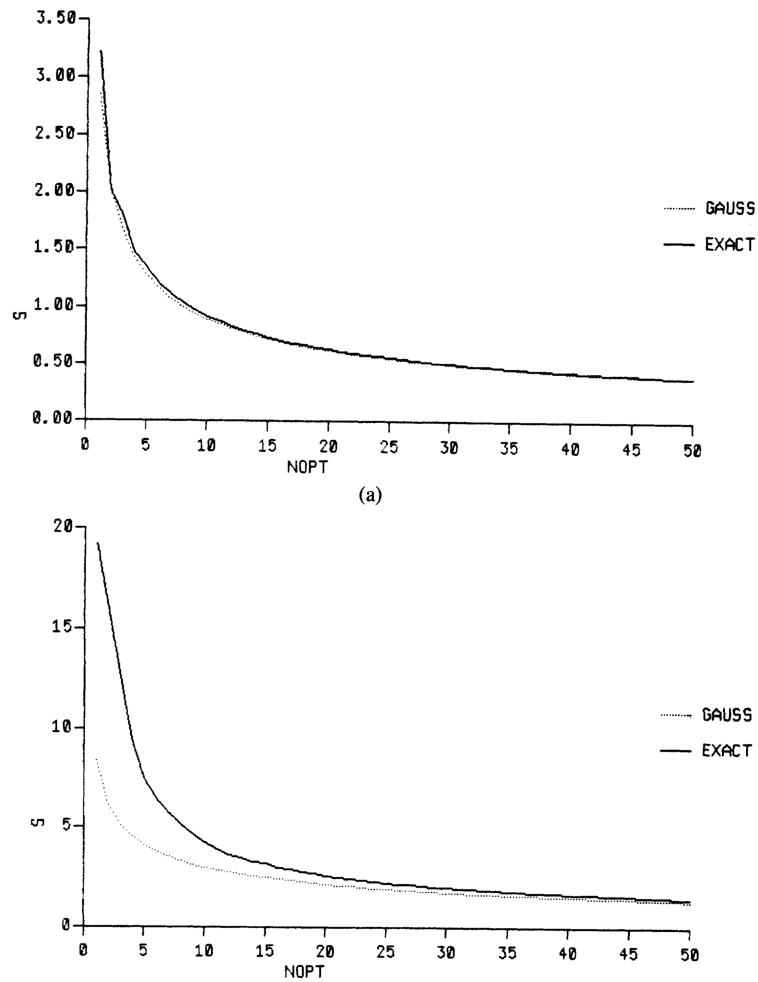


Fig. 5. S Versus N_{opt} under ARE^S.



(b)

Fig. 6. S vesus N_{opt} for small N_{opt}. Fig. 6(a). $\alpha = 0.1$, $\beta = 0.9$. (b) $\alpha = 0.00001$, $\beta = 0.99999$.

of 8/3 for large S. As is to be expected, the value of the ARE at $S = 0$ is 2. Thus, for arbitrarv S

$$N_{\text{lin}} < \frac{8}{3} N_{\text{opt}}.$$

Figs. 10 and 11 show the results for $\text{ARE}^{\alpha}_{\text{opt,sgn}}$ and $\text{ARE}^{\beta}_{\text{opt,sgn}}$, which are plotted by using (39) and (45). Fig. 10 shows that the value of $\text{ARE}^{\alpha}_{\text{opt,sgn}}$ is a rapidly increasing function of S. Fig. 11 shows that the value of $\text{ARE}^{\beta}_{\text{opt,sgn}}$ is actually less than one for $S > 2.26$. Now this is impossible, since the optimal detector is always at least as good as any other detector. The reason for the strange behavior of both $\text{ARE}^{\alpha}_{\text{opt,sgn}}$ and $\text{ARE}^{\beta}_{\text{opt,sgn}}$ for large S is that the Gaussian approximation for the sign

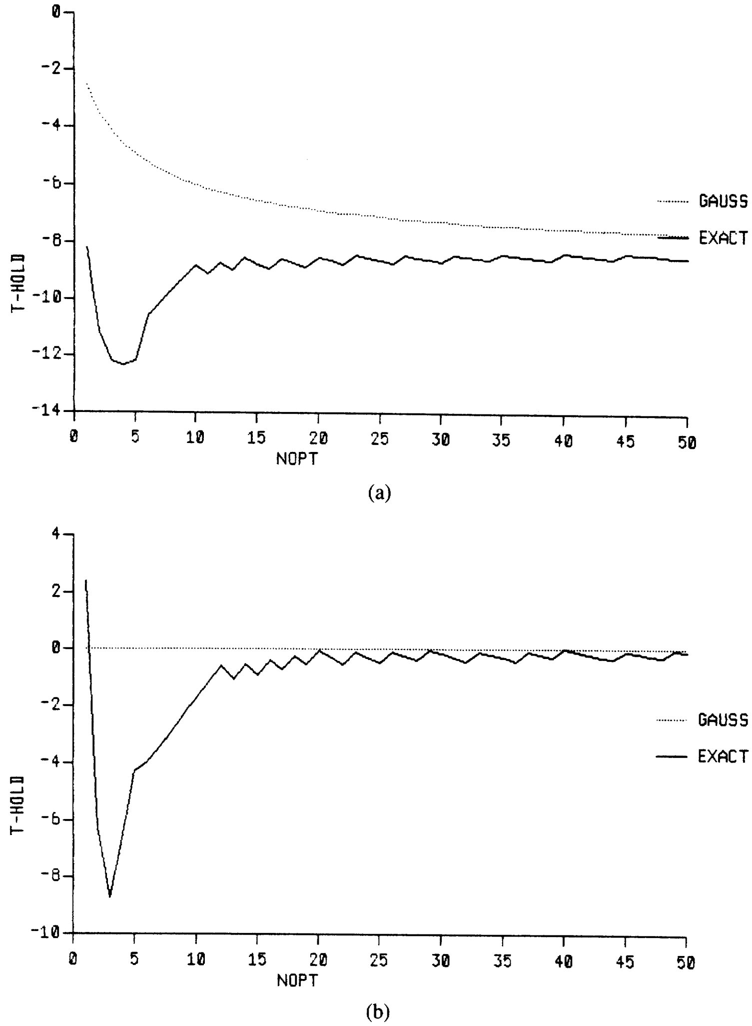


Fig. 7. T_0 versus N_{opt} for small N_{opt}. (a) $\alpha = 0.1$, $\beta = 0.99999$. (b) $\alpha = 0.00001$, $\beta = 0.99999$.

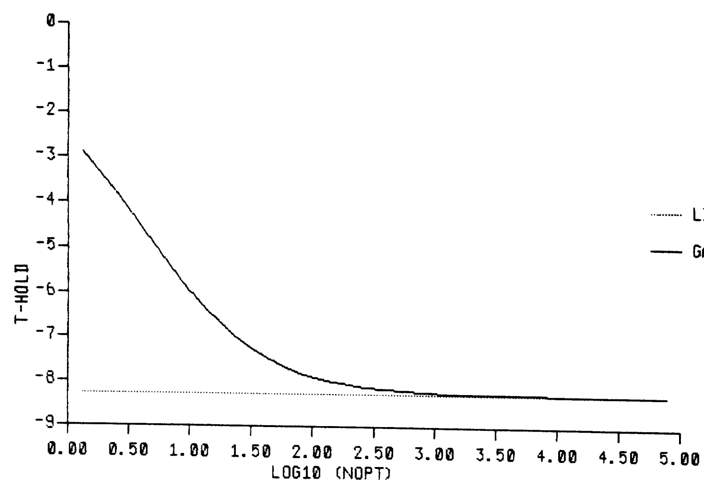


Fig. 8. T_0 Versus N_{opt} for large N_{opt}. $\alpha = 0.1$, $\beta = 0.99999$.

detector is very poor. This is because for large S, the values for α and β are obtained by a summation of very few terms in (23) and (24), i.e., in the Gaussian approximation, the integration of the normal density function is over the tails. However, the Gaussian approximation is very poor over the tails [28, pp. 174–189]. Therefore, the values of $\text{ARE}^{\alpha}_{opt,sgn}$ and $\text{ARE}^{\beta}_{opt,sgn}$ obtained by using the Gaussian approximation are invalid even as an approximation.

Figs. 12(a) through 12(d) show both the exact values and the Gaussian approximation values for $\text{RE}^{\alpha}_{opt,lin}$. The Gaussian approximation is a straight line, which is also

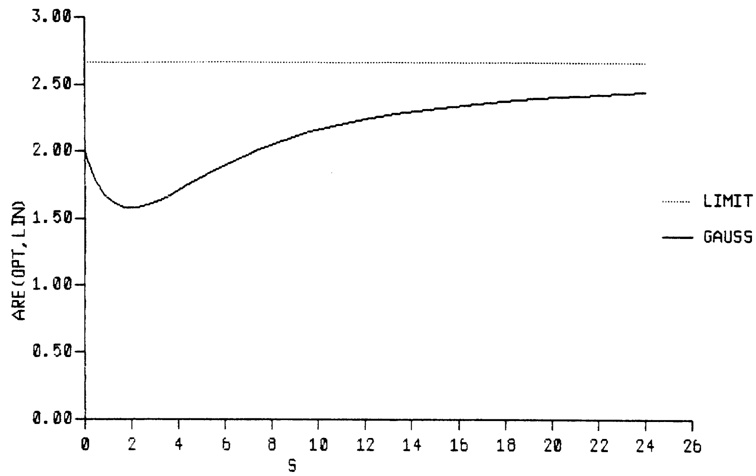


Fig. 9. $ARE^{\alpha,\beta}_{opt,lin}$ versus S.

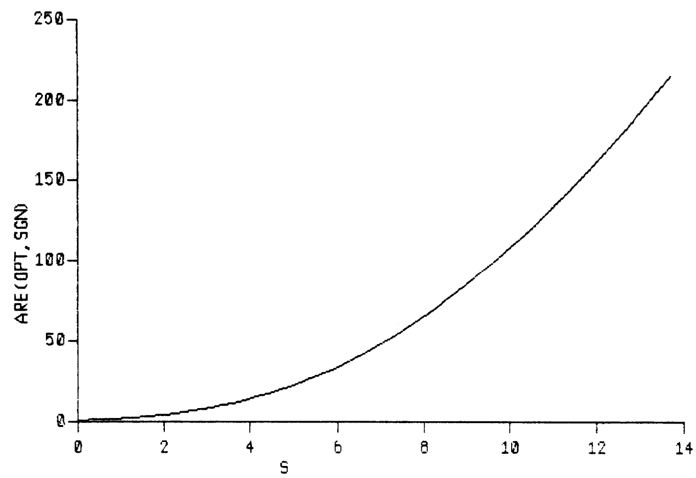


Fig. 10. $ARE^{\alpha}_{opt,sgn}$ versus S.

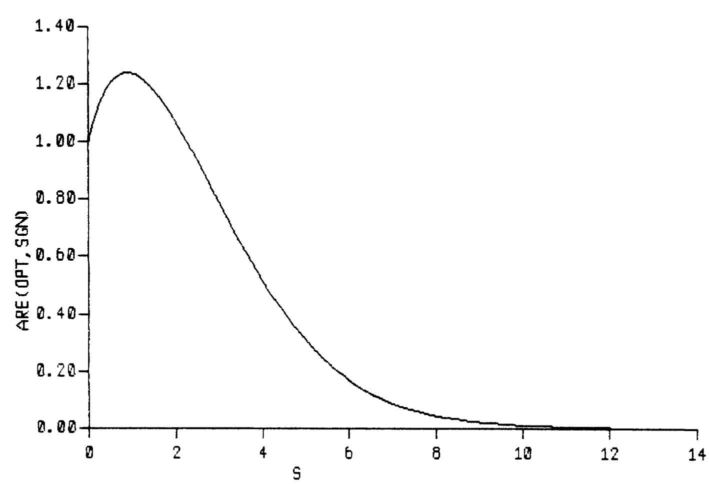


Fig. 11. $ARE^{\beta}_{opt,sgn}$ versus S.

the limit of the exact value as $N_{opt} \rightarrow \infty$. Again, the convergence of the exact value to the Gaussian approximation is seen to be fairly rapid in all cases. It becomes somewhat slower as $\beta \rightarrow 1$.

These observations are also true for $RE^{\beta}_{opt,lin}$. However, instead of $\beta \rightarrow 1$, we have $\alpha \rightarrow 0$, etc.

V. CONCLUSIONS

Analysis of the performance of optimal detectors for moderately large sample sizes is normally intractable. Cases where there are closed form solutions are thus significant in that they provide insight into such performance and illustrate convergence to more easily computed asymptotic values.

In this paper we have taken advantage of the closed-form solution for the optimal detector for a signal in Laplace noise and have presented small sample and asymptotic results for the convergence of the Pitman relative efficiency (RE^S) between the optimal detector and the linear and sign detectors. Results for the dependence of the threshold of the optimal detector and the signal strength on N_{opt} were also derived. In order to present a more complete picture of the detectors' performance, two additional measures (RE^{α} and RE^{β}) and their corresponding asymptotic values were derived. Numerical results were shown to illustrate convergence.

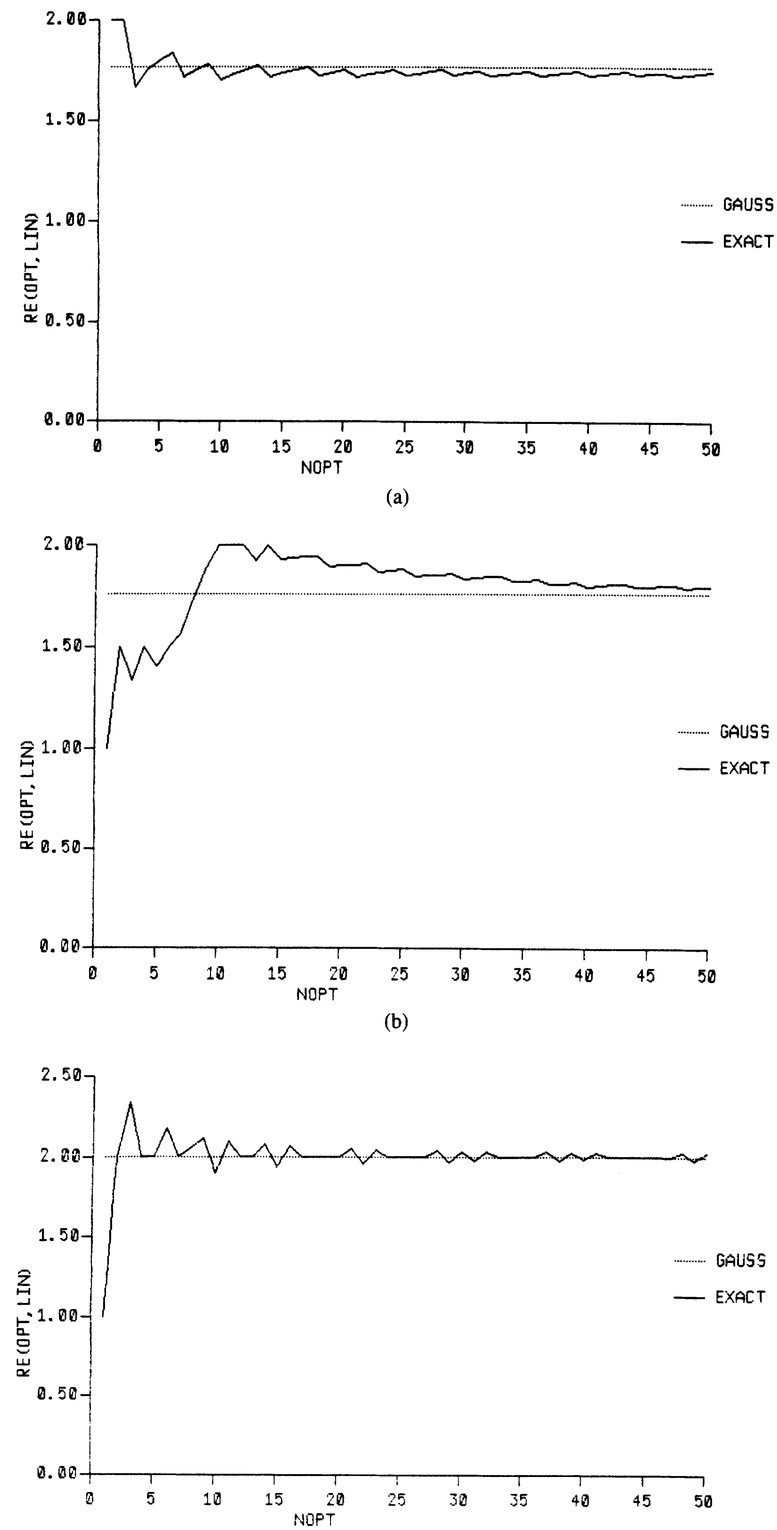
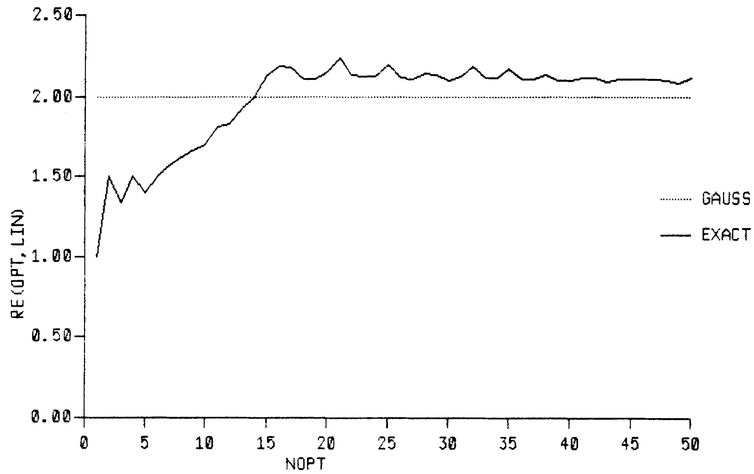


Fig. 12. $RE^{\alpha}_{opt,sgn}$ for small N_{opt}. (a) $S = 0.5$, $\beta = 0.9$. (b) $S = 0.5$, $\beta = 0.99999$. (c) $S = 0.001$, $\beta = 0.9$.



(d)

Fig. 12 (con't.) (d) $S = 0.001$, $\beta = 0.99999$.

APPENDIX
Efficient Evaluation of the Distribution of the Test Statistic t of the Optimal Detector

To evaluate (5) efficiently, we first define

$$y_{j,r} \triangleq \frac{\exp(-r\gamma_L S)}{(j-r)!r!}$$

and

$$\phi_r \triangleq \frac{1}{2\gamma_D}(t + N\gamma_D S) - rS.$$

We note that

$$F_N^{(0)}(t) = \begin{cases} 0, & \text{for } t < -N\gamma_D S \\ 1, & \text{for } t > N\gamma_D S \end{cases}.$$

Also, $k \geq p$ and $N-k \geq q$. Thus

$$\phi_{p+q} \geq \frac{t + N\gamma_D S}{2\gamma_D} - [k + (N-k)]S$$

$$= \frac{t + N\gamma_D S}{2\gamma_D} - NS.$$

If $t \geq N\gamma_D S$, then $F_N^{(0)}(t) = 1$, and $\phi_{p+q} \geq 0$. Similarly, $\phi_m \geq 0$ if $t \geq N\gamma_D S$. For $0 \leq t \leq N\gamma_D S$, it is computationally more efficient to subtract from one a few extra terms rather than sum a larger number of terms in (5). This alternate form can be obtained by subtracting (5) from one and solving for $F_N^{(0)}(t)$

$$F_N^{(0)}(t) = 1 - 2^{-N}N!\Bigg\{\sum_{k=1}^{N}\sum_{p=0}^{k}(-1)^p y_{k,p}\sum_{q=0}^{N-k} y_{N-k,q} \times [1-\exp(-\gamma_L\,\phi_{p+q})e_{k-1} \times (\gamma_L\,\phi_{p+q})]\,[1-u(\phi_{p+q})] + \sum_{m=0}^{N} y_{N,m}\,[1-u(\phi_m)]\Bigg\},$$

$$\text{for } 0 \leq t \leq N\gamma_D S. \quad \text{(A1)}$$

Equation (A1) can be simplified further. We note that $\phi_{p+q} \leq \phi_{N-k+p}$ since $0 \leq q \leq N-k$. Thus we can write

$$[1-u(\phi_{p+q})] = [1-u(\phi_{p+q})][1-u(\phi_{N-k+p})]$$

and (A1) becomes

$$F_N^{(0)}(t) = 1 - 2^{-N}N!\Bigg\{\sum_{k=1}^{N}\sum_{p=0}^{k}(-1)^p y_{k,p} \times [1-u(\phi_{N-k+p})]\sum_{q=0}^{N-k} y_{N-k,q} \times [1-\exp(-\gamma_L\phi_{p+q}) \times e_{k-1}(\gamma_L\phi_{p+q})]\cdot[1-u(\phi_{p+q})] + \sum_{m=0}^{N} y_{N,m}[1-u(\phi_m)]\Bigg\}$$

$$\text{for } 0 \leq t \leq N\gamma_D S. \quad \text{(A2)}$$

Similarly, for $-N\gamma_D S \leq t \leq 0$, we have $\phi_p \geq \phi_{p+q}$. Then

$$u(\phi_{p+q}) = u(\phi_{p+q})u(\phi_p)$$

and

$$F_N^{(0)}(t) = 2^{-N}N!\Bigg\{\sum_{k=1}^{N}\sum_{p=0}^{k}(-1)^p y_{k,p}\,u(\phi_p)\sum_{q=0}^{N-k} \times y_{N-k,q}[1-\exp(-\gamma_L\beta_{p+q}) \times e_{k-1}(\gamma_L\beta_{p+q})]u(\phi_{p+q}) + \sum_{m=0}^{N} y_{N,m}u(\phi_m)\Bigg\},$$

$$\text{for } -N\gamma_D S \leq t \leq 0. \quad \text{(A3)}$$

In computing (A2) or (A3), the unit steps were used to exit from the corresponding summation do-loops. Also, it was found that initial off line establishment of a table of the $y_{j,r}$ and ϕ_r was computationally more efficient than computing them as needed. A Fortran program taking these points into account is given in [12].

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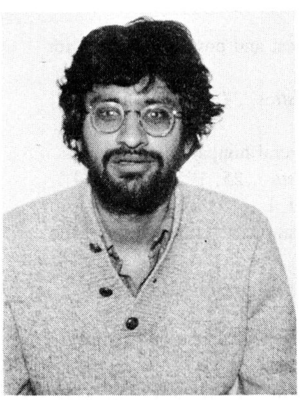

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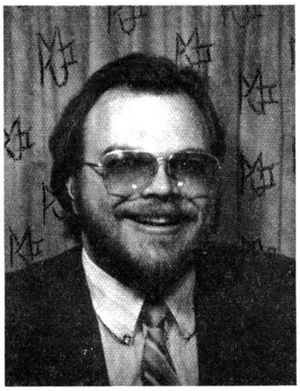

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