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SUPERRESOLUTION VIA ANALYSIS

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A. THE PROBLEM

The source of the superresolution problem is best illustrated by the conventional optical system shown in Figure 1. A coherent plane wave illuminates an object of finite extent, f(x). Thus, incident on the back focal plane is a field amplitude proportional to the Fourier transform of the object:

$$F(u) = \int_{-T}^{T} f(x)e^{-j2\pi ux}dx$$

where the frequency variable u is proportional to the displacement in plane P2. In the pupil plane is a rectangular aperture which passes only those frequency components corresponding to $|u| \le W$. The field amplitude exiting the pupil plane is thus

$$G(u) = F(u)\text{rect}(\frac{u}{2W}) \quad (1)$$

where rect(ξ) is unity for $|\xi| \le \frac{1}{2}$ and is zero otherwise. One form of the superresolution problem is this: given G(u), find F(u). While on the surface it seems that the lost frequency terms are irretrievable, we must remember that, since the object f(x) is of finite extent, F(u) is a bandlimited function. All bandlimited functions are analytic everywhere. Thus, knowledge of the function over any finite interval is sufficient to specify it everywhere. Recall, for example, the Taylor series.

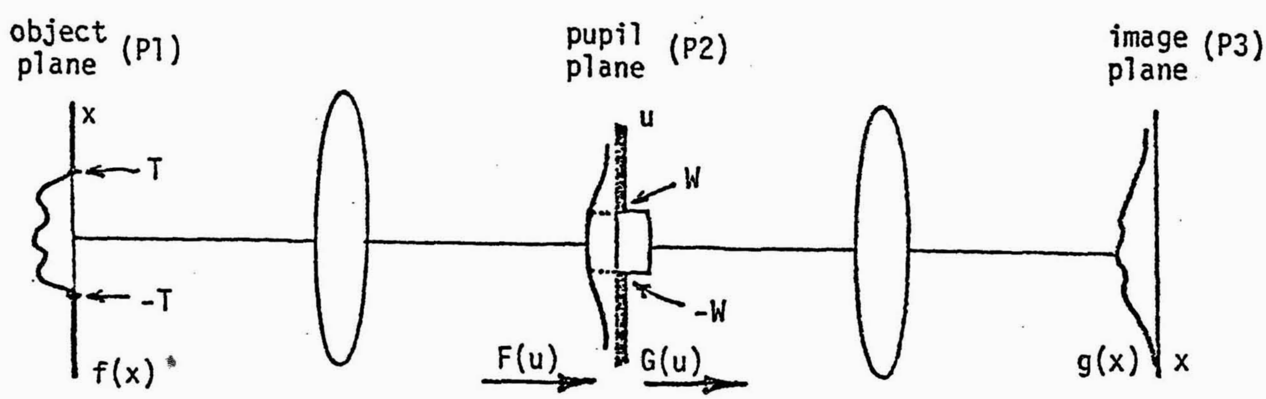

Figure 1: Illustration of a loss of resolution in an imaging system. Each of the three planes is a focal distance from the adjacent lens.

The field amplitude incident on the image plane in Figure 1 is proportional to the inverse transform of Eq. (1) which can be written

$$g(x) = 2W \int_{-T}^{T} f(\xi)\operatorname{sinc} 2W(x-\xi)d\xi \tag{2}$$

where sinc $\xi = \sin(\pi\xi)/(\pi\xi)$. The superresolution problem can thus alternately be stated as follows: Find $f(x)$ with knowledge of $g(x)$ and W.

B. CONTROVERSY

Many superresolution algorithms exist that work exactly in the absence of noise. Unfortunately, the restoration problem, as stated, is ill-posed in the sense that restoration noise level cannot be bound corresponding to an arbitrarily low bound on the input noise level. Upon reflection, this is reasonable. Knowledge of $F(u)$ for $|u| \le W$ should give minimal information about $F(u)$ at, say, $u = 10^8W$. Clearly, however, since $F(u)$ is bandlimited and thus smooth, one should be able to obtain a "good" estimate of $F(u)$ from $G(u)$ at $u = W + \varepsilon$. Global stability should thus be distinguished from local stability.

An ill-posed problem can be "regularized" if we can further constrain the object. Additional information about the object further limits the class of admissable solutions. For example, the object might be known to be real and positive or lie between two bounds. Possibly a bound on the radiant energy of the object is known as a result of the illumination power.

C. EIGEN-FUNCTION APPROACH

One method of solving the (unconstrained) superresolution problem is through solution of the integral equation corresponding to Eq. (2):

$$\lambda_n\psi_n(x) = 2W \int_{-T}^{T} \psi_n(\xi)\operatorname{sinc} 2W(x-\xi)d\xi \tag{3}$$

The eigen-functions $\{\psi_n(x)|n=0,1,2,\cdots\}$ are prolate spheroidal wave functions. The λ_n's are the corresponding eigenvalues. The object can be expanded in the Fourier Series:

$$f(x) = \sum_{n=0}^{\infty} f_n\, \psi_n(x) \tag{4}$$

Using Eq. (2) and Eq. (3), we find that

$$g(x) = \sum_{n=0}^{\infty} \lambda_n f_n\, \psi_n(x)$$

The coefficients $g_n = \lambda_n f_n$ can be found from the image:

$$g_n = \int_{-\infty}^{\infty} g(x)\, \psi_n(x)dx$$

Hence, we can compute $f_n = g_n/\lambda_n$ and, via Eq. (4), the object.

The stability problem in this approach is manifest in the structure of the eigenvalues, λ_n, which are nearly unity from zero to the integer corresponding to the degrees of freedom (space-bandwidth product) of the image. For a greater index, λ_n drops almost to zero. For these values, any uncertainty in measuring g_n is greatly magnified in computation of $f_n = g_n/\lambda_n$.

The prolate spheroidal wave functions are nearly computationally intractible. They serve best as a consise (noiseless) model of superresolution which can be used nicely in proofs of other more tractible linear extrapolation algorithms.

D. ITERATIVE TECHNIQUES

Gerchberg's algorithm is an iterative superresolution technique involving only the operations of Fourier transformation and truncation. Each iteration alternately reinforces the known portion of the spectrum and the bandlimited nature of the function.

The basic algorithm is pictured in Figure 2. We begin in step 1 by inverse transforming the known portion of the spectrum. Since we know the object is of finite extent, we keep only the result for $|x| \leq T$. This is step 2. Step 3 is Fourier transformation. Since we know the answer must be $G(u)$ for $|u| \leq W$, we discard the result over this interval in step 4 and add in $G(u)$ in step 5. This is the first estimate of the superresolved spectrum, $F(u)$. The iteration is continued in step 6 and, in the limit (in the absence of noise) we generate the object.

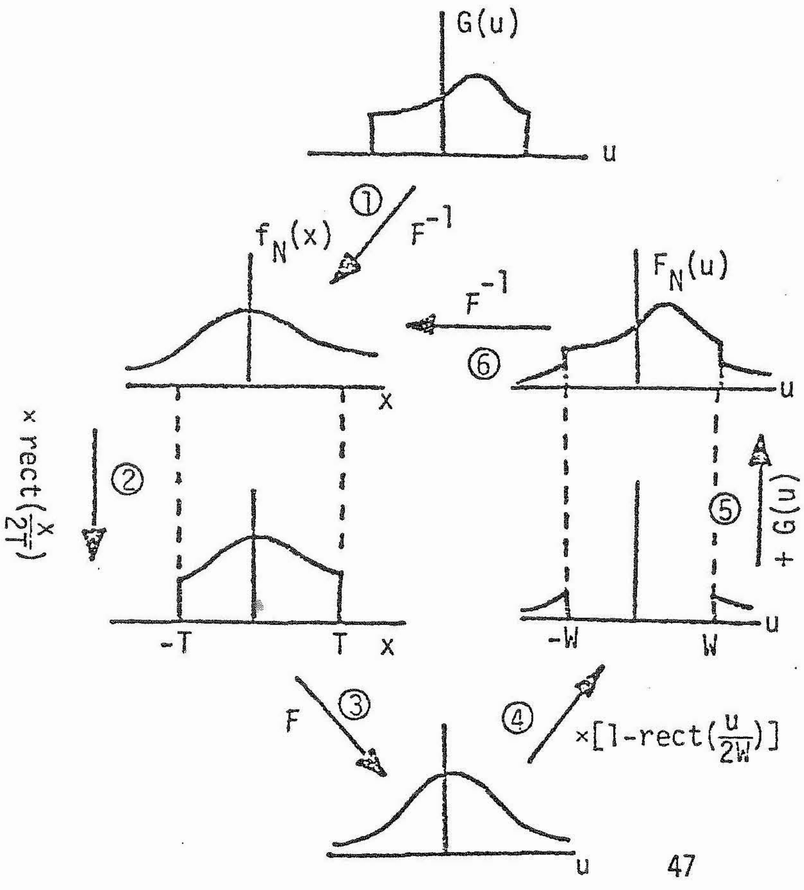

Figure 2: Illustration of Gerchberg's iterative superresolution algorithm. In the absence of noise $F_N \rightarrow F$ and $f_N \rightarrow f$.

In the presence of noise, it has been shown that the restoration noise level bound is proportional to the number of iterations. Hence, as iterations increase, convergence betters and noise worsens. An optimal finite number of iterations is immediately suggested. This number, however, varies widely with the object.

The Gerchberg-type superresolution algorithm is the most flexible of the superresolution algorithms. As we shall see, it is highly adaptive to inclusion of certain constraints.

E. LINEAR NON-RECURSIVE ALGORITHMS

Define the bandlimiting operator

$$\mathcal{B}_T H(u) = 2T \int_{-\infty}^{\infty} H(\eta)\,\text{sinc}\, 2T(u-\eta)\,d\eta$$

Since the object spectrum, F(u), is already a bandlimited function, we have

$$\mathcal{B}_T F(u) = F(u)$$

Hence, we can write Eq. (1) as

$$G(u) = [1-\{1-\text{rect}(\tfrac{u}{2W})\mathcal{B}_T\}]F(u) \tag{5}$$

If we were to simulate this operation digitally, G(u) and F(u) would become sample vectors $\vec{G}$ and $\vec{F}$, $\mathcal{B}_T$ would become a low pass matrix $\underline{B}_T$ and rect $(\frac{u}{2W})$ would become a square matrix $\underline{R}$ with 1's placed in the center of the diagonal, and zero elsewhere. The Eq. (5) becomes

$$\vec{G} = [\underline{I}-\{\underline{I}-\underline{R}\ \underline{B}\}]\vec{F}$$

or

$$\vec{F} = \underline{E}\ \vec{G}$$

where

$$\underline{E} = [\underline{I}-\{\underline{I}-\underline{R}\ \underline{B}\}]^{-1}$$

is the "extrapolation matrix."

Again, in the absense of noise, the algorithm works fine. The extrapolation matrix, however, is ill-conditioned. Small data noise yields enormous restoration error.

F. STATISTICAL MODELS

The models thus far discussed are deterministic. If the object and image are treated stochastically, a wealth of statistical algorithms come into play. Historically, these algorithms were developed for spectral estimation. Most involve finding the solution which extremizes some norm, e.g. minimum mean square error and maximum entropy.

G. FUTURE WORK

There is much current interest in finding techniques to incorporate constraints into superresolution algorithms. The Gerchberg type iterative algorithm seems especially adaptable to this task. There has been success in placing object positivity constraints via "half wave rectification" in each iteration. Similarly, knowledge of a portion of the original object can be blended into the algorithm with ease.

Along the same lines, there is a need for close inspection of the physics of imaging for additional meaningful constraints.

As witnessed by the attached bibliography, there has been a recent explosion of superresolution papers. The need is evident for some unifying performance criterion by which these algorithms can be compared. For the linear algorithms, a possible measure would be the restoration to input noise levels for a given type of noise. It is the non-linearities (such as iterative half-wave rectification in the Gerchberg algorithm), however, that will allow for stable algorithm performance via regularization. In such cases, the restoration noise level is signal dependent.

Lastly, localized superresolution algorithms are in need of inspection. Preliminary work on extending spectra over a larger but finite interval has only recently appeared.

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