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# Stability of an algorithm to restore continuously sampled band-limited images from aliased data

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A continuously sampled object is one periodically set to zero. A continuously sampled band-limited object can be restored by multiplying by an appropriately parameterized periodic function and filtering the product. The sensitivity of this restoration procedure to additive noise is considered. In general, the restoration noise level increases dramatically as the degree of aliasing of the data increases or the duty cycle of the degradation decreases. Numerical results are given for white noise and noise with Laplace autocorrelation. The results are compared with the noise sensitivity of conventional (discrete) Shannon-sampling-theorem interpolation, in which fewer data are used.

## INTRODUCTION

A band-limited object is periodically set to zero. Restoration techniques include conventional linear filtering and logarithmic filtering.[1] (Other possible restoration techniques and corresponding references are given in the introduction of Ref. 2.) A recently presented closed-form algorithm is applicable even when the data are aliased.[2] In this paper the noise sensitivity of this restoration scheme is explored. We demonstrate that, as the severity of aliasing increases or the duty cycle of the degradation decreases, the effects of additive noise on the image yields an ever-increasing noise level. Numerical examples are offered for white noise and noise with Laplace autocorrelation. The results are compared with conventional (discrete) Shannon-sampling-theorem interpolation, in which fewer data are used.

## PRELIMINARIES

Let $f(x)$ denote a finite-energy band-limited signal with bandwidth $2W$. That is,

$$f(x) = \int_{-W}^{W} \mathrm{F}(u)\exp(j2\pi ux)du,$$

where

$$\mathrm{F}(u) = \int_{-\infty}^{\infty} f(x)\exp(-j2\pi ux)dx$$
$$= \mathcal{F}f(x)$$

and $\mathcal{F}$ denotes the Fourier-transform operation. Define a unit period pulse train with duty cycle $\alpha < 1$:

$$r_\alpha(x) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{x-n}{\alpha}\right),$$

where rect$(y)$ is unity for $|y| \leq 1/2$ and is zero otherwise. The continuously sampled image, illustrated in Fig. 1, is

$$g(x) = f(x)r_\alpha(x/T),$$

where $T$ is a specified period. The restoration problem is to regain $f(x)$ from knowledge of $g(x)$ and $2W$.

A linear restoration scheme, schematically depicted in Fig. 2, is[2]

$$f(x) = [g(x)\psi_M(x/T)] * 2W\,\mathrm{sinc}(2Wx), \tag{1}$$

where the asterisk denotes convolution and $\mathrm{sinc}(y) = \sin(\pi y)/(\pi y)$. The periodic function $\psi_M(x)$ is defined by

$$\psi_M(x) = \theta_M(x)r_\alpha(x), \tag{2}$$

where $\theta_M(x)$ is the trigonometric polynomial

$$\theta_m(x) = \sum_{m=-M}^{M} b_m \exp(-j2\pi mx) \tag{3}$$

and the $b_m$'s are the solution to the Toeplitz set of equations

$$\sum_{m=-M}^{M} b_m c_{n-m} = \delta_n, \qquad |n| \leq M. \tag{4}$$

Here, $\delta_n$ denotes the Kronecker delta function and $c_n = \alpha$ sinc $\alpha n$. $M$ is equal to the greatest integer not exceeding $2WT$ and is called the degree of aliasing of the degradation.

By using Eq. (3), Eq. (2) can be written alternatively in Fourier-series form as

$$\psi_M(x) = \sum_{m=-\infty}^{\infty} d_n \exp(-j2\pi nx), \tag{5}$$

where

$$d_n = \begin{cases} \delta_n, & |n| \leq M \\ \sum_{m=-M}^{M} b_m c_{n-m}, & |n| > M \end{cases}. \tag{6}$$

## EFFECTS OF NOISE ON THE RESTORATION SCHEME

We consider here the effects of additive wide-sense-stationary zero-mean noise, $\xi(x)$, superimposed upon $g(x)$. Because of linearity, an input of $g(x) + \xi(x)$ into the restoration algorithm will yield an output of $f(x) + \eta(x)$, where $\eta(x)$ is the algorithm

 

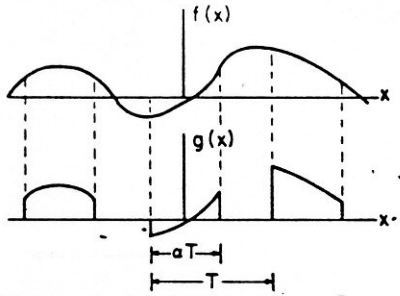


Fig. 1. Illustration of a band-limited object $f(x)$ and the corresponding continuously sampled $g(x)$.

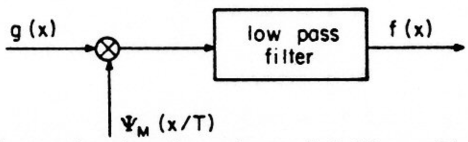


Fig. 2. Restoration of continuously sampled objects. The periodic function $\psi_M$ is parameterized by the duty cycle $\alpha$ and the severity of aliasing. The low-pass filter has the same bandwidth as the restored signal.

response to $\xi(x)$ alone. Using Eq. (1):

$$\eta(x) = [\xi(x)\psi_M(x/T)] * 2W \operatorname{sinc} 2Wx.$$

In general, $\eta(x)$ is also zero mean, although it is not stationary.

The restoration noise level follows as

$$\begin{aligned}\overline{\eta^2(x)} &= E[\eta^2(x)] \\ &= E\left\{\int_{\tau=-\infty}^{\infty} \xi(\tau)\psi_M\left(\frac{\tau}{T}\right) 2W \operatorname{sinc}[2W(x-\tau)]d\tau \right. \\ &\quad \left. \times \int_{\lambda=-\infty}^{\infty} \xi(\lambda)\psi_M\left(\frac{\lambda}{T}\right) 2W \operatorname{sinc}[2W(x-\lambda)]d\lambda\right\} \\ &= 4W^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_\xi(\tau-\lambda)\psi_M\left(\frac{\tau}{T}\right)\psi_M\left(\frac{\lambda}{T}\right) \\ &\quad \times \operatorname{sinc}[2W(x-\tau)]\operatorname{sinc}[2W(x-\lambda)]d\tau d\lambda,\end{aligned}$$

where $E$ denotes the expected value operator and the input noise autocorrelation is

$$R_\xi(\tau - \lambda) = E[\xi(\tau)\xi(\lambda)].$$

By straightforward integral manipulations we have

$$\overline{\eta^2(x)} = \int_{-\infty}^{\infty} R_\xi(\lambda)h(x;\lambda)d\lambda, \tag{7}$$

where

$$\begin{aligned}h(x;\lambda) &= 2W\psi_M\left(\frac{\lambda}{T}\right)\operatorname{sinc} 2W(x-\lambda) \\ &\quad \star\, 2W\psi_M\left(\frac{\lambda}{T}\right)\operatorname{sinc} 2W(x-\lambda).\end{aligned} \tag{8}$$

The star denotes autocorrelation with respect to $\lambda$. The output noise level in Eq. (7) is an even periodic function with period $T$.

## White Noise

For white noise

$$R_\xi(\tau) = \overline{\xi^2}\delta(\tau). \tag{9}$$

Equation (7) becomes

$$\begin{aligned}\overline{\eta^2(x)}/\overline{\xi^2} &= h(x;0) \\ &= (2W)^2 \operatorname{sinc}^2(2Wx) * \psi^2{}_M\left(\frac{x}{T}\right).\end{aligned} \tag{10}$$

From Eq. (2),

$$\psi^2{}_M(x) = r_\alpha(x)\theta^2{}_M(x),$$

where, from Eq. (3),

$$\theta^2{}_M(x) = \sum_{|k|\le M}\sum_{|l|\le M} b_k b_l \exp[-j2\pi(k+l)x]. \tag{11}$$

Fourier transforming both sides of Eq. (10) gives

$$\begin{aligned}\mathcal{F}\overline{\eta^2(x)}/\overline{\xi^2} &= 2W\Lambda\left(\frac{u}{2W}\right)\mathcal{F}\psi^2{}_m\left(\frac{x}{T}\right) \\ &= 2W\Lambda\left(\frac{u}{2W}\right)[TR_\alpha(Tu) * \mathcal{F}\theta^2{}_M(x/T)],\end{aligned} \tag{12}$$

where

$$\Lambda(y) = (1 - |y|)\operatorname{rect}(y/2)$$

and

$$\begin{aligned}R_\alpha(u) &= \mathcal{F}r_\alpha(x) \\ &= \sum_{p=-\infty}^{\infty} c_p\delta(u+p)\end{aligned}$$

and $\delta(y)$ is the Dirac delta function. Substituting this and the transform of Eq. (11) into Eq. (12) gives

$$\begin{aligned}\mathcal{F}\overline{\eta^2}/\overline{\xi^2} &= 2W\Lambda\left(\frac{u}{2W}\right) \\ &\quad \times \left[\sum_{p=-\infty}^{\infty} c_p \sum_{|k|\le M} b_k \sum_{|\ell|\le M} b_\ell \delta\left(u + \frac{k+\ell-p}{T}\right)\right] \\ &= 2W\Lambda\left(\frac{u}{2W}\right)\sum_{|k|\le M} b_k \\ &\quad \times \sum_{|\ell|\le M} b_\ell \sum_{|q|\le M} c_{k+\ell-q}\delta\left(u - \frac{q}{T}\right),\end{aligned}$$

where $q = k + \ell - p$ and we have recognized that the finite extent of the triangle function lets through only $2M + 1$ of the Dirac delta functions. Evaluating $\Lambda(u/2W)$ at $u = q/T$ and inverse transforming gives the desired result:

$$\begin{aligned}\overline{\eta^2(x)}/\overline{\xi^2} &= \sum_{|k|\le M} b_k \sum_{|\ell|\le M} b_\ell \\ &\quad \times \sum_{|q|\le M}\left(2W - \frac{|q|}{T}\right) c_{k+\ell-q}\exp(-j2\pi qx/T).\end{aligned} \tag{13}$$

An illustration of the restoration noise level for various duty cycles for first-degree aliasing is shown in Fig. 3. The effects of variation of aliasing order are illustrated in Fig. 4.

## Colored Noise

With the aim of placing Eq. (7) in more tractable form for colored noise, we Fourier transform with respect to $\lambda$ using the autocorrelation theorem of Fourier analysis[3]:

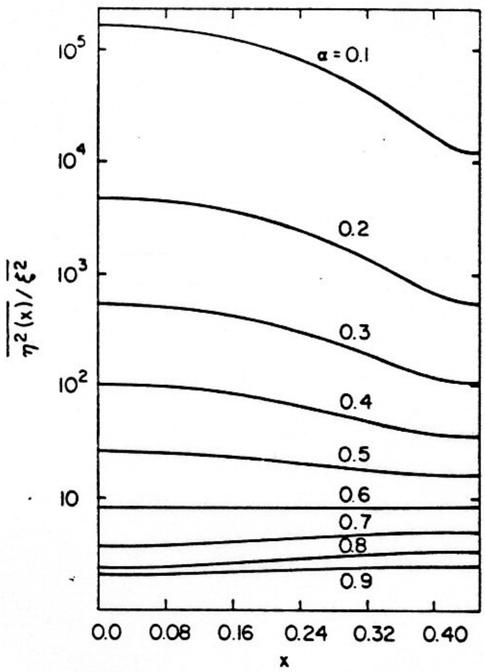


Fig. 3. Normalized restoration noise level for additive white noise for various duty cycles $\alpha$. $2W = 2$ and $T = 0.9$, giving $M$ = first-order aliasing.

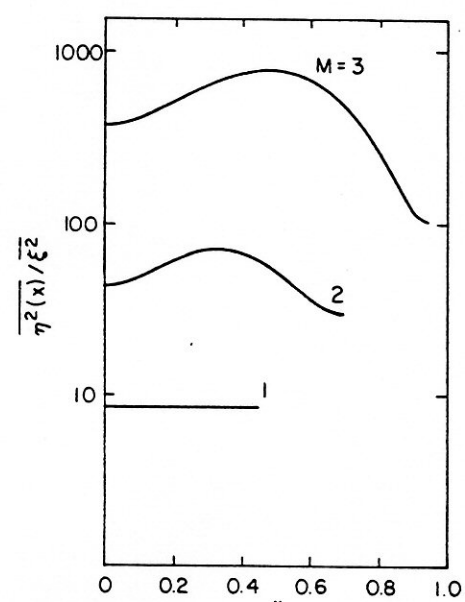


Fig. 4. Normalized restoration noise level for additive white noise for various orders of aliasing $M$. The values of $T$ corresponding to $M = 1, 2, 3$ are $T = 0.9, 1.4, 1.9$, respectively. Because of symmetry, plots are needed only for $0 \le x \le T/2$. ($2W = 2$ and $\alpha = 0.6$.)

$$H(x; \upsilon) = \int_{-\infty}^{\infty} h(x; \lambda) \exp(-j2\pi\upsilon\lambda) d\lambda$$

$$= \left| T\Psi_M(T\upsilon) * \left[\exp(-j2\pi\upsilon x)\mathrm{rect}\left(\frac{\upsilon}{2W}\right)\right]\right|^2 \qquad (14)$$

where $\Psi_M(\upsilon)$ is the Fourier transform of $\psi_M(x)$ and convolution is with respect to $\upsilon$. From Eq. (5):

$$\Psi_M(\upsilon) = \sum_{n=-\infty}^{\infty} d_n \delta(\upsilon + n).$$

Thus Eq. (14) becomes

$$H(x; \upsilon) = \left| \sum_{n=-\infty}^{\infty} d_n \exp\left[-j2\pi x\left(\upsilon + \frac{n}{T}\right)\right] \mathrm{rect}\left(\frac{\upsilon + \frac{n}{T}}{2W}\right)\right|^2$$

$$= \left| \sum_{n=-\infty}^{\infty} d_n \exp(-j2\pi nx/T) \mathrm{rect}\left(\frac{\upsilon + \frac{n}{T}}{2W}\right)\right|^2. \qquad (15)$$

Since

$$\mathrm{rect}\left(\frac{\upsilon + \frac{n}{T}}{2W}\right)\mathrm{rect}\left(\frac{\upsilon + \frac{m}{T}}{2W}\right)$$

$$= \mathrm{rect}\left(\frac{m-n}{4WT}\right)\mathrm{rect}\left(\frac{\upsilon + \frac{m+n}{2T}}{2W - \frac{|n-m|}{T}}\right),$$

substituting into Eq. (15) and further recognizing from Eq. (6) that $d_m = \delta_m$ for $|m| \le M$ gives

$$H(x; \upsilon) = \mathrm{rect}\left(\frac{\upsilon}{2W}\right) + \sum_{|n|>M} \sum_{\substack{|m|>M \\ |n-m|\le M}} d_n d_m$$

$$\times\, \mathrm{rect}\left(\frac{\upsilon + \frac{m+n}{2T}}{2W - \frac{|m-n|}{T}}\right) \exp[-j2\pi(n-m)x/T]$$

$$= \mathrm{rect}\left(\frac{\upsilon}{2W}\right) + \sum_{|n|>M} \sum_{\substack{|m|\le M \\ |n-m|>M}} d_n d_{n-m}$$

$$\times\, \mathrm{rect}\left(\frac{\upsilon + \frac{2n-m}{2T}}{2W - \frac{|m|}{T}}\right) \exp(-j2\pi mx/T). \qquad (16)$$

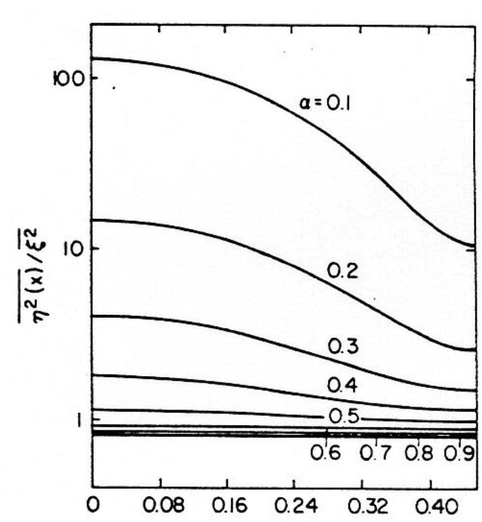


Fig. 5. Normalized restoration noise level for input noise with Laplace autocorrelation for various duty cycles $\alpha$. $2W = 2$ and $T = 0.9$, giving $M$ = first-degree aliasing. The Laplace parameter is $a = 2$.

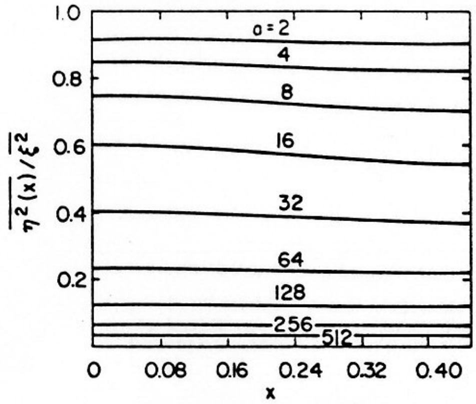


Fig. 6. Normalized restoration noise level for additive input noise with Laplace autocorrelation for various Laplace parameters $a$. $2W = 2$ and $T = 0.9$, giving $M$ = first-degree aliasing. The duty cycle is $\alpha = 0.6$.

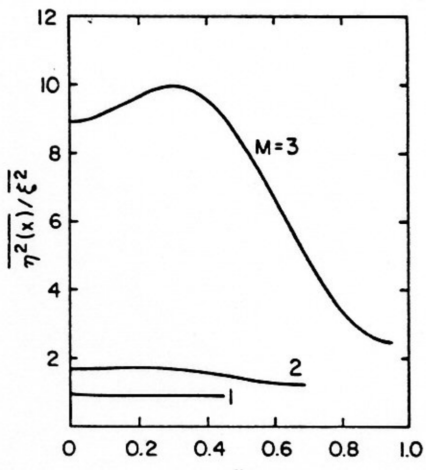


Fig. 7. Normalized restoration noise level for additive input noise with Laplace autocorrelation for various degrees of aliasing. The $T$ values corresponding to $M = 1, 2, 3$ are $T = 0.9, 1.4, 1.9$, respectively. Because of symmetry, plots are needed only for $0 \le x \le T/2$. ($2W = 2$, $\alpha = 0.6$, and $a = 2$.)

By using Parseval's formula,[4] Eq. (7) can be written as

$$\overline{\eta^2(x)} = \int_{-\infty}^{\infty} S_\xi(\upsilon) H(x; \upsilon) d\upsilon, \qquad (17)$$

where the power spectral density $S_\xi(\upsilon)$ is the Fourier transform of $R_\xi(\lambda)$. Define the odd indefinite integral

$$I_\xi(u) = \int_0^u S_\xi(\upsilon) d\upsilon.$$

Then substituting Eq. (16) into Eq. (17) and recognizing $\overline{\eta^2(x)}$ is real gives the Fourier series

$$\overline{\eta^2(x)} = 2I_\xi(W) + \sum_{|n|>M} \sum_{\substack{|m|\le M\\ |n-m|>M}} d_n d_{n-m} \times \cos(2\pi m x/T)\left[I_\xi\left(\frac{m-|m|-2n}{2T}+W\right) - I_\xi\left(\frac{m+|m|-2n}{2T}-W\right)\right]. \qquad (18)$$

For white noise, as in Eq. (9), $I_\xi(u) = \overline{\xi^2} u$. The equivalent result in Eq. (13), however, is in closed form.

For an example application of Eq. (18), consider the Laplace autocorrelation

$$R_\xi(\tau) = \overline{\xi^2}\, e^{-a|\tau|},$$

where $a > 0$ is a specified parameter. Then

$$I_\xi(u) = \frac{\overline{\xi^2}}{\pi} \arctan\left(\frac{2\pi u}{a}\right).$$

In the numerical examples to follow, $W$ is set to unity. Figure 5 shows the dependence of output noise level on the duty cycle $\alpha$ for first-order aliasing. The dependence of the Laplace parameter is shown in Fig. 6 for a fixed duty cycle. As $a$ increases, adjacent points of the input noise become less correlated and the interpolation noise level decreases. Dependence of the output noise level on order of aliasing is illustrated in Fig. 7.

Some observations follow: (1) Clearly, the restoration noise level increases dramatically with an increase in the degree of aliasing $M$ or a decrease of the duty cycle $\alpha$. The condition of the Toeplitz matrix[5] corresponding to Eq. (4) is equivalently worsened. (2) For $M$th-order aliased data, $\psi_N(x)$ can be used in Fig. 2 in lieu of $\psi_M(x)$ if $N \ge M$. Our results, however, clearly demonstrate that the use of higher-order $\psi$'s to restore lower-order aliased data significantly increases the restoration noise level.

## COMPARISON WITH SAMPLING THEOREM RESTORATION

For certain combinations of the parameters $T$, $2W$, and $\alpha$, the continuously sampled signal can be discretely sampled uniformly at or in excess of the Nyquist rate. Let this rate be denoted by $2B \ge 2W$. The result is the same as if we had discretely sampled the original signal at a rate $2B$.

Let $T$, $2W$, and $\alpha$ be such that this uniform sampling can be performed. Assume that, as in the previous section, each sample point is perturbed by additive Laplace autocorrelation noise with parameter $a$. When the noisy samples are inter-

Table 1. Noise Level Comparison[a]

| Example | $T$ | $2W$ | $\alpha$ | $2B$ | min $\overline{\eta^2(x)}/\overline{\xi^2}$ | max $\overline{\eta^2(x)}/\overline{\xi^2}$ | $\overline{\eta_0{}^2}/\overline{\xi^2}$ |
|---|---|---|---|---|---|---|---|
| (a) | 1 | 1.5 | 0.9 | 5 | 0.7460 | 0.7478 | 0.7647 |
| (b) | 1 | 1.5 | 0.7 | 3 | 0.7768 | 0.7842 | 0.8020 |
| (c) | 5 | 0.3 | 0.96 | 1 | 0.2810 | 0.2822 | 0.3754 |
| (d) | 5 | 0.2 | 0.2 | 0.2 | 19.925 | 19.925 | 1.000 |

[a] Comparison of noise levels for some cases in which the signal can be restored using either the continuously sampled signal-restoration algorithm or the conventional sampling theorem (followed by filtering). The former, in each case but one, yields a better noise level. In (d), for which this is not the case, $M = 0$ restoration can be used even though, as in each entry, $M = \langle 2WT \rangle = 1$. In each case, the Laplace parameter is $a = 2$.

polated and passed through a filter unity on $|u| \leq W$ and zero elsewhere, the output noise level is[6]

$$\overline{\eta_o{}^2}/\overline{\xi^2} = \frac{2}{\pi} \arctan \left[ \frac{\sinh\left(\frac{a}{2B}\right)\tan\left(\frac{\pi W}{2B}\right)}{\cosh\left(\frac{a}{2B}\right) - 1} \right]. \tag{19}$$

As one would expect, fewer data are used in restoration, and thus a higher noise level results.

Four examples of noise levels are shown in Table 1. For each entry, $a = 2$ and $M = 1$. In order to fit the uniform sampling within the continuously sampled intervals, each case requires that sampling be done such that there is a sample at the origin. In each case but one, the noise level in Eq. (19) exceeds the maximum restoration level in Eq. (18) for the same noise. For example (d) in Table 1, this is not the case. Note, however, that even though $M = 2WT = 1$, the spectra are simply touching with no overlap. Hence no unscrambling of aliased spectra is required. Rather, for restoration, we should here restore using $M = 0$.

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